Dispersion and Attenuation of Love Waves in a Stack of N Isotropic Viscoelastic Layers over a Half-Space: A Thomson–Haskell Propagator Matrix Approach

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1 Conventions

Every sign in this work follows from the choices below, so we fix them once and use them throughout.

- **Fourier transform pair.** For a function of time $f(t)$,

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt, \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{+i\omega t}\, d\omega.$$

  Consequently $\partial_t \mapsto i\omega$ and the implied harmonic time factor of every field below is $e^{+i\omega t}$.

- **Horizontal dependence.** A wave travelling in the $+x$ direction therefore carries the factor e^{-ikx}, and the full phase factor is

$$e^{i(\omega t - kx)}, \qquad \mathrm{Re}(k) > 0.$$

- **Depth.** z increases downwards. The free surface is $z = z_0 = 0$; interfaces are at $z_1 < z_2 < \cdots < z_N$; layer j occupies $z_{j-1} \le z \le z_j$ and has thickness $h_j = z_j - z_{j-1}$. The half-space is layer $N+1$ and occupies $z \ge z_N$.

- **Attenuation.** With the factor $e^{i(\omega t - kx)}$, a wave decaying in the direction of propagation has $\mathrm{Im}(k) < 0$; the spatial attenuation coefficient is $\alpha = -\,\mathrm{Im}(k) > 0$. Likewise, a complex modulus $\tilde{\mu} = \mu' + i\mu''$ is dissipative when $\mu'' > 0$, and $Q^{-1} = \mu''/\mu'$.

- **Symbols reserved.** μ' and μ'' always denote the real and imaginary parts of a complex modulus. The subscripted μ_0, μ_u and μ_R denote the *relaxed*, *unrelaxed* and (equivalently to μ_0) *relaxed* moduli of the viscoelastic model; μ_j is the complex shear modulus of layer j; and μ_l is the spring modulus of the l-th Maxwell body. Context disambiguates the last two, which never appear together.

The elastic case is recovered by letting all imaginary parts vanish. The viscoelastic results are obtained from the elastic ones by the **correspondence principle**: solve the elastic problem in the frequency domain and replace every real modulus μ by the complex modulus $\tilde{\mu}(\omega)$. This is legitimate because the equation of motion is independent of the constitutive law, and the constitutive law is a multiplication in the frequency domain.

2 Isotropic Elastic case

The 3D equations of motion for an isotropic linear-elastic medium can be written as:

$$\begin{aligned}
\rho\frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x \\
\rho\frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y \\
\rho\frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z
\end{aligned} \tag{1}$$

The tensor of elastic moduli for an **isotropic medium** is given as:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{bmatrix}$$

$$
\begin{aligned}
\sigma_{xx} &= \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{xx} \\
\sigma_{yy} &= \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{yy} \\
\sigma_{zz} &= \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{zz} \\
\sigma_{xy} &= 2\mu\epsilon_{xy} \\
\sigma_{yz} &= 2\mu\epsilon_{yz} \\
\sigma_{zx} &= 2\mu\epsilon_{zx}
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\epsilon_{xx} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x}\right) = \frac{\partial u_x}{\partial x} &\qquad \epsilon_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\
\epsilon_{yy} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y}\right) = \frac{\partial u_y}{\partial y} &\qquad \epsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\
\epsilon_{zz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z}\right) = \frac{\partial u_z}{\partial z} &\qquad \epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)
\end{aligned}
$$

Equation (2) can be written generally as:

$$\boxed{\sigma_{ij} = \lambda\vartheta\delta_{ij} + 2\mu\epsilon_{ij}}$$

where:

- λ and μ are the Lamé parameters (elastic constants)
- $\vartheta = \epsilon_{kk} = \nabla \cdot \mathbf{u}$ is the dilatation (volumetric strain)
- δ_{ij} is the Kronecker delta
- $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the strain tensor
- u_i the displacement [m]
- σ_{ij} the stress tensor [Pa]
- f_i the source term [N/m^3]
- ρ the density [kg/m^3]

Throughout we abbreviate the Cartesian displacement components as $u \equiv u_x$, $v \equiv u_y$, $w \equiv u_z$.

3 Isotropic ViscoElastic

Now, to describe a viscoelastic medium, we need to modify the stress-strain relation because the conservation of momentum is independent of the material behavior. In linear viscoelasticity **the stress depends on the history of the strain rate**. The viscoelastic stress-strain relation can be described by generalizing the purely elastic case by **introducing frequency-dependent complex moduli (or quality factor, Q)** or **time-domain convolution integrals described by the Boltzmann superposition and causality principle**:

$$\sigma(t) = \int_{-\infty}^{t} \Psi(t-\tau)\dot{\epsilon}(\tau)\, d\tau$$

$\Psi(t)$ is the relaxation function.

$$\boxed{\sigma_{ij}(t) = \int_{-\infty}^{t} \Psi_{ijkl}(t-\tau)\dot{\epsilon}_{kl}(\tau)\, d\tau}$$

where $\Psi_{ijkl}(t)$ is the **relaxation tensor** and ε_{kl} is the infinitesimal strain,

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right).$$

Using the time convolution notation:

$$a(t) * b(t) = \int_{-\infty}^{\infty} a(t-\tau)\, b(\tau)\, d\tau \overset{\text{causal}}{=} \int_{0}^{t} a(t-\tau)\, b(\tau)\, d\tau,$$

where the second equality holds only when both a and b vanish for negative argument. The relaxation tensor is causal by construction, $\Psi_{ijkl}(t) \propto H(t)$, which is what guarantees that the stress at time t depends only on the strain history up to t.
The above can be written compactly as:

$$\boxed{\sigma_{ij} = \Psi_{ijkl} * \dot{\varepsilon}_{kl}} \tag{3}$$

For an isotropic viscoelastic medium, the constitutive relation takes the form:

$$\boxed{\sigma_{ij} = \delta_{ij}\,(\Psi_\lambda * \dot{\vartheta}) + 2\,\Psi_\mu * \dot{\varepsilon}_{ij}} \tag{4}$$

4 Viscoelastic Stress Components

From (4), we calculate the stress components:

Normal Stresses ($\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$)

$$\sigma_{xx}(t) = \Psi_\lambda(t) * \dot{\vartheta}(t) + 2\Psi_\mu(t) * \left(\frac{\partial \dot{u}(t)}{\partial x}\right)$$
$$\sigma_{yy}(t) = \Psi_\lambda(t) * \dot{\vartheta}(t) + 2\Psi_\mu(t) * \left(\frac{\partial \dot{v}(t)}{\partial y}\right)$$
$$\sigma_{zz}(t) = \Psi_\lambda(t) * \dot{\vartheta}(t) + 2\Psi_\mu(t) * \left(\frac{\partial \dot{w}(t)}{\partial z}\right)$$

Shear Stresses ($\sigma_{xy}, \sigma_{yz}, \sigma_{xz}$)

$$\sigma_{xy}(t) = 2\Psi_\mu(t) * \left[\frac{1}{2}\left(\frac{\partial \dot{u}(t)}{\partial y} + \frac{\partial \dot{v}(t)}{\partial x}\right)\right] = \Psi_\mu(t) * \left(\frac{\partial \dot{u}(t)}{\partial y} + \frac{\partial \dot{v}(t)}{\partial x}\right)$$

$$\sigma_{yz}(t) = \Psi_\mu(t) * \left(\frac{\partial \dot{v}(t)}{\partial z} + \frac{\partial \dot{w}(t)}{\partial y}\right)$$
$$\sigma_{xz}(t) = \Psi_\mu(t) * \left(\frac{\partial \dot{u}(t)}{\partial z} + \frac{\partial \dot{w}(t)}{\partial x}\right)$$

Summary and Final Notes:

Assembling all the components, we arrive at the complete equation:

$$\begin{aligned}
\sigma_{xx} &= \Psi_\lambda * \dot{\vartheta} + 2\Psi_\mu * \frac{\partial \dot{u}}{\partial x} & \qquad \sigma_{xy} &= \Psi_\mu * \left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right) \\
\sigma_{yy} &= \Psi_\lambda * \dot{\vartheta} + 2\Psi_\mu * \frac{\partial \dot{v}}{\partial y} & \qquad \sigma_{yz} &= \Psi_\mu * \left(\frac{\partial \dot{v}}{\partial z} + \frac{\partial \dot{w}}{\partial y}\right) \\
\sigma_{zz} &= \Psi_\lambda * \dot{\vartheta} + 2\Psi_\mu * \frac{\partial \dot{w}}{\partial z} & \qquad \sigma_{xz} &= \Psi_\mu * \left(\frac{\partial \dot{u}}{\partial z} + \frac{\partial \dot{w}}{\partial x}\right)
\end{aligned}$$

Physical Meaning of the Relaxation Functions:

- $\Psi_\mu(t)$: The **shear relaxation modulus**. It describes the time-dependent stress response to a step change in shear strain. It controls the dissipation of S-waves.
- $\Psi_\lambda(t)$: This function, along with $\Psi_\mu(t)$, governs the relaxation of volumetric stress. It controls the dissipation of P-waves.

In the frequency domain, these convolutions become simple multiplications, and the complex moduli derived from $\tilde{\Psi}_\lambda(\omega)$ and $\tilde{\Psi}_\mu(\omega)$ define the frequency-dependent velocities and attenuation (quality factors Q_P and Q_S) of the medium.

The constitutive equations in the frequency domain become

$$\tilde{\sigma}_{xx} = i\omega\tilde{\Psi}_\lambda\tilde{\vartheta} + 2i\omega\tilde{\Psi}_\mu\frac{\partial\tilde{u}}{\partial x}, \qquad \tilde{\sigma}_{xy} = i\omega\tilde{\Psi}_\mu\left(\frac{\partial\tilde{u}}{\partial y} + \frac{\partial\tilde{v}}{\partial x}\right),$$
$$\tilde{\sigma}_{yy} = i\omega\tilde{\Psi}_\lambda\tilde{\vartheta} + 2i\omega\tilde{\Psi}_\mu\frac{\partial\tilde{v}}{\partial y}, \qquad \tilde{\sigma}_{yz} = i\omega\tilde{\Psi}_\mu\left(\frac{\partial\tilde{v}}{\partial z} + \frac{\partial\tilde{w}}{\partial y}\right),$$
$$\tilde{\sigma}_{zz} = i\omega\tilde{\Psi}_\lambda\tilde{\vartheta} + 2i\omega\tilde{\Psi}_\mu\frac{\partial\tilde{w}}{\partial z}, \qquad \tilde{\sigma}_{xz} = i\omega\tilde{\Psi}_\mu\left(\frac{\partial\tilde{u}}{\partial z} + \frac{\partial\tilde{w}}{\partial x}\right).$$

$$\tilde{\sigma}_{xx} = \lambda(\omega)\tilde{\vartheta} + 2\mu(\omega)\frac{\partial\tilde{u}}{\partial x}, \qquad \tilde{\sigma}_{xy} = \mu(\omega)\left(\frac{\partial\tilde{u}}{\partial y} + \frac{\partial\tilde{v}}{\partial x}\right),$$
$$\tilde{\sigma}_{yy} = \lambda(\omega)\tilde{\vartheta} + 2\mu(\omega)\frac{\partial\tilde{v}}{\partial y}, \qquad \tilde{\sigma}_{yz} = \mu(\omega)\left(\frac{\partial\tilde{v}}{\partial z} + \frac{\partial\tilde{w}}{\partial y}\right),$$
$$\tilde{\sigma}_{zz} = \lambda(\omega)\tilde{\vartheta} + 2\mu(\omega)\frac{\partial\tilde{w}}{\partial z}, \qquad \tilde{\sigma}_{xz} = \mu(\omega)\left(\frac{\partial\tilde{u}}{\partial z} + \frac{\partial\tilde{w}}{\partial x}\right).$$

where the complex moduli are given by:

$$\lambda(\omega) = i\omega\,\tilde{\Psi}_\lambda(\omega) = \int_{0^-}^{\infty} \dot{\Psi}_\lambda(t)\,e^{-i\omega t}\,dt \tag{5}$$

$$\mu(\omega) = i\omega\,\tilde{\Psi}_\mu(\omega) = \int_{0^-}^{\infty} \dot{\Psi}_\mu(t)\,e^{-i\omega t}\,dt \tag{6}$$

The integrand is the time derivative of the relaxation function, evaluated at t; the lower limit 0^- is required because $\dot{\Psi}$ contains a Dirac delta at the origin (the relaxation function jumps from 0 to its unrelaxed value there). The two forms are equivalent by the derivative theorem, $\mathcal{F}[\dot{\Psi}] = i\omega\,\tilde{\Psi}$, valid in the distributional sense for causal Ψ.

2D SH wave in ViscoElastic Media

SH Wave Configuration

For SH (Shear Horizontal) waves:

1. Particle motion in y-direction
2. Propagation in x-direction
3. Variation in z-direction

4. Only non-zero displacement: $u_y(x,z,t)$

5. Only non-zero stresses: σ_{xy}, σ_{zy}

So, constitutive relations are:

$$\sigma_{xy} = \Psi_\mu * \left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right) = \Psi_\mu * \frac{\partial \dot{v}}{\partial x}$$

$$\sigma_{yz} = \Psi_\mu * \left(\frac{\partial \dot{v}}{\partial z} + \frac{\partial \dot{w}}{\partial y}\right) = \Psi_\mu * \frac{\partial \dot{v}}{\partial z}$$

In Frequency Domain (Fourier Transform)

$$\tilde{\sigma}_{xy} = \mu(\omega)\left(\frac{\partial \tilde{v}}{\partial x}\right)$$

$$\tilde{\sigma}_{yz} = \mu(\omega)\left(\frac{\partial \tilde{v}}{\partial z}\right)$$

Complex Shear Modulus

The complex shear modulus can be expressed as:

$$\tilde{\mu}(\omega) = \mu'(\omega) + i\,\mu''(\omega), \qquad \mu'' > 0 \text{ for } \omega > 0.$$

A concrete realization is the **Maxwell model** (a spring of modulus μ in series with a dashpot of viscosity η). Elements in series share the stress and add their strains, so $1/\tilde{\mu} = 1/\mu + 1/(i\omega\eta)$, giving

$$\tilde{\mu}(\omega) = \frac{i\mu\omega\eta}{i\omega\eta + \mu}$$

where:

- μ: shear modulus of the spring
- η: dashpot viscosity

This single element cannot produce a frequency-independent Q; the Generalized Maxwell body assembled later remedies that.

5 Wave Equation in Viscoelastic Media

Equation of Motion

Retaining only the y-component of the momentum equation and setting the body force $f_y = 0$ (we seek free oscillations of the layered medium, not a forced response):

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} = \rho \frac{\partial^2 u_y}{\partial t^2}$$

Fourier transforming in time ($\partial_t^2 \mapsto -\omega^2$) and substituting the viscoelastic constitutive relations:

$$\frac{\partial}{\partial x}\left(\tilde{\mu}(\omega)\frac{\partial \tilde{u}_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\tilde{\mu}(\omega)\frac{\partial \tilde{u}_y}{\partial z}\right) = -\rho\omega^2 \tilde{u}_y$$

For a homogeneous viscoelastic medium $\tilde{\mu}$ is independent of position, so it comes outside the derivatives:

$$\boxed{\tilde{\mu}(\omega)\left(\frac{\partial^2 \tilde{u}_y}{\partial x^2} + \frac{\partial^2 \tilde{u}_y}{\partial z^2}\right) = -\rho\omega^2 \tilde{u}_y}$$

6 Plane Wave Solution

Following the convention $e^{i(\omega t - kx)}$ fixed at the outset, separate variables:

$$\tilde{u}_y(x, z, \omega) = A(z)\, e^{-ikx}$$

Compute derivatives:

$$\frac{\partial^2 \tilde{u}_y}{\partial x^2} = -k^2 A(z) e^{-ikx}, \qquad \frac{\partial^2 \tilde{u}_y}{\partial z^2} = A''(z) e^{-ikx}.$$

Substituting into

$$\tilde{\mu}(\omega)\left(\frac{\partial^2 \tilde{u}_y}{\partial x^2} + \frac{\partial^2 \tilde{u}_y}{\partial z^2}\right) = -\rho\omega^2 \tilde{u}_y,$$

we obtain

$$\tilde{\mu}(\omega)\big(A''(z) - k^2 A(z)\big)e^{-ikx} = -\rho\omega^2 A(z) e^{-ikx}$$

Dividing both sides by e^{-ikx} gives

$$\tilde{\mu}(\omega)\big(A'' - k^2 A\big) = -\rho\omega^2 A.$$

Simplifying,

$$A'' + \left(\frac{\rho\omega^2}{\tilde{\mu}(\omega)} - k^2\right) A = 0.$$

Let

$$\nu^2 := \frac{\omega^2}{\beta^2(\omega)} - k^2, \qquad \nu = \sqrt{\frac{\omega^2}{\beta^2(\omega)} - k^2}, \qquad \beta(\omega) = \sqrt{\frac{\tilde{\mu}(\omega)}{\rho}}$$

with the branch Re $\beta(\omega) > 0$ so that β reduces to the ordinary shear speed in the elastic limit. In a viscoelastic medium $\tilde{\mu}$, and hence ν, is complex, so no real/imaginary case distinction is available. The general solution valid for *any* complex ν is

$$A(z) = C_1 e^{i\nu z} + C_2 e^{-i\nu z} \qquad \Longleftrightarrow \qquad A(z) = C_1' \cos(\nu z) + C_2' \sin(\nu z),$$

the two forms being related by $C_1' = C_1 + C_2$, $C_2' = i(C_1 - C_2)$. The elastic special cases – oscillatory when $\nu^2 > 0$ and evanescent when $\nu^2 < 0$ (whereupon $\nu = -i\gamma$ and $e^{-i\nu z} = e^{-\gamma z}$) – are contained in this single expression. Hence

$$\tilde{u}_y(x, z, \omega) = \big(A\, e^{ik\, r_\beta z} + B\, e^{-ik\, r_\beta z}\big)\, e^{-ikx}, \qquad \nu = k\, r_\beta,$$

$$r_\beta = \sqrt{\frac{c^2}{\beta^2(\omega)} - 1}, \qquad c = \frac{\omega}{k}.$$

We use ν in preference to q or kr_β from here on.

7 General Solution for Love Wave in ViscoElastic Medium

For Love waves in a homogeneous layer, the general solution is:

$$\tilde{u}_y(x, z, \omega) = \big(Ae^{i\nu z} + Be^{-i\nu z}\big)\, e^{-ikx}, \qquad \nu = k\, r_\beta, \qquad r_\beta = \sqrt{\frac{c^2}{\beta^2(\omega)} - 1}$$

$$\tilde{\sigma}_{yz} = \tilde{\mu}(\omega) \left(\frac{\partial \tilde{u}_y}{\partial z}\right)$$

$$\boxed{\tilde{\sigma}_{yz} = i\tilde{\mu}\,\nu\left(Ae^{i\nu z} - Be^{-i\nu z}\right)e^{-ikx}.}$$

Stripping the common factor e^{-ikx} (which is continuous across every horizontal interface and therefore plays no role in the boundary conditions), we write the z-dependent part in matrix form as a **state vector**:

$$\begin{bmatrix} \tilde{u}_y(z) \\ \tilde{\sigma}_{yz}(z) \end{bmatrix} = \begin{bmatrix} e^{i\nu z} & e^{-i\nu z} \\ i\tilde{\mu}\nu\, e^{i\nu z} & -i\tilde{\mu}\nu\, e^{-i\nu z} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \tag{7}$$

8 Thomson–Haskell Propagator Matrix Method for Love Wave Dispersion Analysis

The **Thomson–Haskell propagator matrix method** is a frequency-domain method for plane waves in a multilayered half-space, introduced by Thomson [8] and corrected and applied to surface-wave dispersion by Haskell [6]. It provides a layer-by-layer solution for body-wave propagation and surface-wave dispersion problems, and is the standard approach for multi-layered media [1]. Now, let's reformulate this problem using propagator matrices.

The Core Problem and Idea for Love Waves

Problem: Calculate the dispersion and attenuation of Love waves propagating in a stack of N horizontal, viscoelastic layers over a semi-infinite viscoelastic half-space.

Core Idea (Propagator Matrix): The state of the SH wavefield at any depth z is described by a **State Vector** (7) containing the relevant continuous field quantities (displacement and shear stress). The **Layer Propagator Matrix** relates the state vector at the top of a layer to its value at the bottom. By successively propagating the state vector from the half-space to the free surface and applying the appropriate boundary conditions, the dispersion equation for the layered medium can be derived.

The Foundation: State Vector and Field Matrix for SH Waves

For SH waves, the motion is purely in the y-direction (transverse to the propagation direction x and depth z).

The **State Vector**, $\mathbf{f}(z)$, for SH waves is:

$$\mathbf{f}(z) = \begin{bmatrix} u_y(z) \\ \\ \sigma_{yz}(z) \end{bmatrix}$$

Where:

- $u_y(z)$: Amplitude of horizontal displacement.
- $\sigma_{yz}(z)$: Shear stress component.

The general solution within a homogeneous, isotropic layer j is a superposition of upgoing and downgoing plane waves:

$$u_y^{(j)}(z) = A_j e^{i\nu_j(z-z_{j-1})} + B_j e^{-i\nu_j(z-z_{j-1})}$$

$$\sigma_{yz}^{(j)}(z) = \mu_j \frac{\partial u_y^{(j)}}{\partial z} = i\mu_j\nu_j \left(A_j e^{i\nu_j(z-z_{j-1})} - B_j e^{-i\nu_j(z-z_{j-1})}\right)$$

where:

- $\nu_j = k\sqrt{(c/\beta_j)^2 - 1}$ is the vertical wavenumber in layer j (can be real or complex).
- A_j is the amplitude of the **upgoing** wave.
- B_j is the amplitude of the **downgoing** wave.
- μ_j is the complex shear modulus of layer j (incorporating viscoelasticity).
- $k = \omega/c$ is the horizontal wavenumber.

We define the **Amplitude Vector**, $\mathbf{a}_j$:

$$\mathbf{a}_j = \begin{bmatrix} A_j \\ B_j \end{bmatrix}$$

The mathematical link between the state vector and the amplitude vector is given by the **Field Matrix**, $\mathbf{E}_j(z)$:

$$\mathbf{f}(z) = \mathbf{E}_j(z)\mathbf{a}_j$$

Explicitly, this is:

$$\begin{bmatrix} u_y(z) \\ \sigma_{yz}(z) \end{bmatrix} = \begin{bmatrix} e^{i\nu_j(z-z_{j-1})} & e^{-i\nu_j(z-z_{j-1})} \\ i\mu_j\nu_j e^{i\nu_j(z-z_{j-1})} & -i\mu_j\nu_j e^{-i\nu_j(z-z_{j-1})} \end{bmatrix} \begin{bmatrix} A_j \\ B_j \end{bmatrix}$$

Derivation of the Layer Propagator Matrix for SH Waves

We want to relate the state vector at the top of a layer ($z = z_t = z_{j-1}$) to the state vector at the bottom ($z = z_b = z_j$).

1. **State at the Bottom:** $\mathbf{f}_{\text{bottom}} = \mathbf{E}_j(z_b)\mathbf{a}_j$. We can solve for the amplitude vector:

$$\mathbf{a}_j = \mathbf{E}_j^{-1}(z_b)\mathbf{f}_{\text{bottom}}$$

   The inverse of the 2×2 field matrix is straightforward to compute:

$$\mathbf{E}_j^{-1}(z) = \frac{1}{2}\begin{bmatrix} e^{-i\nu_j(z-z_{j-1})} & -\frac{i}{\mu_j\nu_j}e^{-i\nu_j(z-z_{j-1})} \\ e^{i\nu_j(z-z_{j-1})} & \frac{i}{\mu_j\nu_j}e^{i\nu_j(z-z_{j-1})} \end{bmatrix}$$

2. **State at the Top:** $\mathbf{f}_{\text{top}} = \mathbf{E}_j(z_t)\mathbf{a}_j$.

3. **Connect Top to Bottom:** Substitute the expression for $\mathbf{a}_j$:

$$\mathbf{f}_{\text{top}} = \mathbf{E}_j(z_t)\left[\mathbf{E}_j^{-1}(z_b)\mathbf{f}_{\text{bottom}}\right] = \underbrace{\mathbf{E}_j(z_t)\mathbf{E}_j^{-1}(z_b)}_{\mathbf{T}_j}\mathbf{f}_{\text{bottom}}$$

   We define the **Layer Propagator Matrix, $\mathbf{T}_j$**:

$$\mathbf{T}_j = \mathbf{E}_j(z_t)\mathbf{E}_j^{-1}(z_b)$$

Let's compute this explicitly. Set the local coordinate so the top of the layer is at $z' = 0$ and the bottom is at $z' = h_j$. Thus $z_t = 0$, $z_b = h_j$.

$$\mathbf{E}_j(0) = \begin{bmatrix} 1 & 1 \\ i\mu_j\nu_j & -i\mu_j\nu_j \end{bmatrix}$$

$$\mathbf{E}_j(h_j) = \begin{bmatrix} e^{i\nu_j h_j} & e^{-i\nu_j h_j} \\ i\mu_j\nu_j e^{i\nu_j h_j} & -i\mu_j\nu_j e^{-i\nu_j h_j} \end{bmatrix}$$

Form the product $\mathbf{T}_j = \mathbf{E}_j(0)\mathbf{E}_j^{-1}(h_j)$, using Euler's formulas

$$\cos x = \tfrac{1}{2}\left(e^{ix} + e^{-ix}\right), \qquad \sin x = \tfrac{1}{2i}\left(e^{ix} - e^{-ix}\right).$$

Writing $\theta_j = \nu_j h_j$, the entries are

$$
\begin{aligned}
(\mathbf{T}_j)_{11} &= \tfrac{1}{2}\left(e^{-i\theta_j} + e^{i\theta_j}\right) = \cos\theta_j, \\
(\mathbf{T}_j)_{12} &= \frac{i}{2\mu_j\nu_j}\left(e^{i\theta_j} - e^{-i\theta_j}\right) = \frac{i}{\mu_j\nu_j}\left(i\sin\theta_j\right) = -\frac{\sin\theta_j}{\mu_j\nu_j}, \\
(\mathbf{T}_j)_{21} &= \frac{i\mu_j\nu_j}{2}\left(e^{-i\theta_j} - e^{i\theta_j}\right) = \frac{i\mu_j\nu_j}{2}\left(-2i\sin\theta_j\right) = \mu_j\nu_j\sin\theta_j, \\
(\mathbf{T}_j)_{22} &= \tfrac{1}{2}\left(e^{-i\theta_j} + e^{i\theta_j}\right) = \cos\theta_j.
\end{aligned}
$$

Hence

$$
\boxed{\mathbf{T}_j = \begin{bmatrix} \cos(\nu_j h_j) & -\dfrac{\sin(\nu_j h_j)}{\mu_j\nu_j} \\ \mu_j\nu_j\sin(\nu_j h_j) & \cos(\nu_j h_j) \end{bmatrix}}
$$

This is the **Thomson–Haskell propagator matrix** for Love waves.

Physical Meaning of $\mathbf{T}_j$: This matrix is a property of the layer alone. If we know the displacement and stress at the bottom, we obtain them at the top by multiplying by $\mathbf{T}_j$. It "propagates" the SH solution *upwards* through the layer.

Three checks.

1. **Consistency with the first-order system.** Combining $\sigma_{yz} = \mu_j\,\partial_z u_y$ with the equation of motion (in which $\partial_x^2 \mapsto -k^2$ and $\partial_t^2 \mapsto -\omega^2$), the state vector obeys

$$
\frac{d}{dz}\begin{bmatrix} u_y \\ \sigma_{yz} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1/\mu_j \\ k^2\mu_j - \rho_j\omega^2 & 0 \end{bmatrix}}_{\mathbf{A}_j} \begin{bmatrix} u_y \\ \sigma_{yz} \end{bmatrix}, \qquad k^2\mu_j - \rho_j\omega^2 = -\mu_j\nu_j^2.
$$

   The *downward* propagator over a thickness h_j is $\exp(\mathbf{A}_j h_j)$, and one verifies directly that $\mathbf{T}_j = \exp(-\mathbf{A}_j h_j) = [\exp(\mathbf{A}_j h_j)]^{-1}$, as it must be, since $\mathbf{T}_j$ propagates upward. A useful diagnostic: if the off-diagonal signs of $\mathbf{T}_j$ are flipped, one obtains $\exp(+\mathbf{A}_j h_j)$, i.e. the *downward* propagator, and the chain rule below is then applied backwards.

2. **Unimodularity.** $\det\mathbf{T}_j = \cos^2\theta_j + \sin^2\theta_j = 1$ for every complex θ_j. This holds because $\operatorname{tr}\mathbf{A}_j = 0$, and it survives the product: $\det\mathbf{G} = 1$. It is a cheap and sharp numerical test.

3. **Branch independence.** $\mathbf{T}_j$ depends on ν_j only through $\cos(\nu_j h_j)$ and $\sin(\nu_j h_j)/\nu_j$,

both of which are *even* functions of ν_j. The sign of the square root defining ν_j is therefore irrelevant inside the layers. Only the half-space vertical wavenumber ν_{N+1} requires a branch choice, because there the radiation condition singles out one exponential.

Building for General N-Layer System

Consider N layers over a half-space (layer $N+1$). The free surface is $z_0 = 0$, the interfaces are at depths $z_1 < z_2 < \cdots < z_N$, and the thickness of layer j is $h_j = z_j - z_{j-1}$.

At each interface, the state vector is continuous (welded contact):

$$\mathbf{f}^{(j)}_{\text{bottom}} = \mathbf{f}^{(j+1)}_{\text{top}}$$

Using the propagator matrix for each layer:

$$\mathbf{f}^{(j)}_{\text{top}} = \mathbf{T}_j \mathbf{f}^{(j)}_{\text{bottom}} = \mathbf{T}_j \mathbf{f}^{(j+1)}_{\text{top}}$$

We can chain these relations from the half-space up to the top layer:

$$\mathbf{f}^{(1)}_{\text{top}} = \mathbf{T}_1 \mathbf{f}^{(2)}_{\text{top}} = \mathbf{T}_1 \mathbf{T}_2 \mathbf{f}^{(3)}_{\text{top}} = \cdots = \mathbf{T}_1 \mathbf{T}_2 \cdots \mathbf{T}_N \, \mathbf{f}^{(N+1)}_{\text{top}}$$

Let us define the **Global Propagator Matrix**, $\mathbf{G}$:

$$\mathbf{G} = \mathbf{T}_1 \mathbf{T}_2 \cdots \mathbf{T}_N = \prod_{j=1}^{N} \mathbf{T}_j$$

(the product is ordered, j increasing to the right, and does not commute). So the final relationship is:

$$\boxed{\mathbf{f}_{\text{surface}} = \mathbf{G}\, \mathbf{f}_{\text{halfspace-top}}}$$

Where:

- $\mathbf{f}_{\text{surface}} = \mathbf{f}^{(1)}_{\text{top}}$ is the state vector at the free surface ($z = 0$).
- $\mathbf{f}_{\text{halfspace-top}} = \mathbf{f}^{(N+1)}_{\text{top}}$ is the state vector at the top of the half-space, $z = z_N$.

Since each $\det \mathbf{T}_j = 1$, we also have $\det \mathbf{G} = 1$.

Applying Boundary Conditions and Finding Love Waves

Boundary Condition 1: Free Surface ($z = 0$)

At the free surface, the shear stress is zero.

$$\mathbf{f}_{\text{surface}} = \begin{bmatrix} u_y(0) \\ 0 \end{bmatrix}$$

Boundary Condition 2: Radiation Condition in the Half-Space ($z \geq z_N$)

In the half-space (layer $N+1$) the solution must be purely **downgoing**; there can be no energy returning from infinity, so the amplitude of the upgoing wave is $A_{N+1} = 0$. For a trapped Love mode it must in addition be **evanescent**, i.e. decay as $z \to \infty$.

The general solution in the half-space, in the local coordinate $z - z_N$, is:

$$\tilde{u}_y^{(N+1)}(z) = \left(A_{N+1}e^{i\nu_{N+1}(z-z_N)} + B_{N+1}e^{-i\nu_{N+1}(z-z_N)}\right)e^{-ikx}$$

$$\tilde{\sigma}_{yz}^{(N+1)}(z) = i\mu_{N+1}\nu_{N+1}\left(A_{N+1}e^{i\nu_{N+1}(z-z_N)} - B_{N+1}e^{-i\nu_{N+1}(z-z_N)}\right)e^{-ikx}$$

Since $A_{N+1} = 0$, and dropping the common e^{-ikx}:

$$\tilde{u}_y^{(N+1)}(z) = B_{N+1}\,e^{-i\nu_{N+1}(z-z_N)}$$

$$\tilde{\sigma}_{yz}^{(N+1)}(z) = -i\mu_{N+1}\nu_{N+1}B_{N+1}e^{-i\nu_{N+1}(z-z_N)}$$

Branch of ν_{N+1}. Write $\nu_{N+1} = \nu_R + i\nu_I$. Then

$$\left|e^{-i\nu_{N+1}(z-z_N)}\right| = e^{\nu_I\,(z-z_N)},$$

so the retained solution decays as $z \to \infty$ *if and only if*

$$\boxed{\mathrm{Im}(\nu_{N+1}) < 0}$$

This is the correct branch for the form written above. For the elastic case with $c < \beta_{N+1}$ one has $\nu_{N+1}^2 < 0$, so $\nu_{N+1} = -i\gamma_{N+1}$ with $\gamma_{N+1} = k\sqrt{1 - c^2/\beta_{N+1}^2} > 0$, and indeed $\tilde{u}_y \propto e^{-\gamma_{N+1}(z-z_N)}$.

Why the branch cannot be chosen carelessly. Taking the opposite branch while retaining the $e^{-i\nu_{N+1}(z-z_N)}$ form replaces $-i\mu_{N+1}\nu_{N+1}$ by $+i\mu_{N+1}\nu_{N+1}$ in the boundary vector $\mathbf{V}$ and the dispersion function D (both defined below), changing D from $G_{21} - i\mu_{N+1}\nu_{N+1}G_{22}$ to $G_{21} + i\mu_{N+1}\nu_{N+1}G_{22}$. These are *not* proportional, so the two have *different zeros*: the wrong branch yields spurious modes, not merely a sign change. For the single layer over a half-space it replaces the correct $\tan(\nu_1 h_1) = \mu_2\gamma_2/(\mu_1\nu_1)$ by $\tan(\nu_1 h_1) = -\mu_2\gamma_2/(\mu_1\nu_1)$.

There is one – and only one – combination of errors that hides itself. If the off-diagonal signs of $\mathbf{T}_j$ are *also* flipped, then $\mathbf{G}$ is replaced by $\mathbf{JGJ}$ with $\mathbf{J} = \mathrm{diag}(1, -1)$, so $G_{21} \mapsto -G_{21}$ and $G_{22} \mapsto G_{22}$; combined with the flipped branch this sends $D \mapsto -D$, leaving the roots untouched. A code carrying *both* errors therefore produces correct dispersion curves while producing half-space eigenfunctions that grow exponentially with depth. Neither error alone is benign, and the pair is benign only for the roots.

Therefore, at the top of the half-space ($z = z_N$), the state vector is:

$$\mathbf{f}_{\text{halfspace-top}} = \begin{bmatrix} 1 \\ -i\mu_{N+1}\nu_{N+1} \end{bmatrix} B_{N+1}$$

We can write this as:

$$\mathbf{f}_{\text{halfspace-top}} = \mathbf{V}\, B_{N+1}, \quad \text{where} \quad \mathbf{V} = \begin{bmatrix} 1 \\ -i\mu_{N+1}\nu_{N+1} \end{bmatrix}$$

Here, $\mathbf{V}$ is the **boundary vector** for the half-space.

Formulating the Dispersion Equation

Substitute the half-space condition into the **global propagation** relation:

$$\mathbf{f}_{\text{surface}} = \mathbf{G}\,\mathbf{f}_{\text{halfspace-top}} = \mathbf{G}\,\mathbf{V}\, B_{N+1}$$

Write this out:

$$\begin{bmatrix} u_y(0) \\ 0 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -i\mu_{N+1}\nu_{N+1} \end{bmatrix} B_{N+1} = \begin{bmatrix} G_{11} - i\mu_{N+1}\nu_{N+1}G_{12} \\ G_{21} - i\mu_{N+1}\nu_{N+1}G_{22} \end{bmatrix} B_{N+1}$$

This gives us two equations:

1. $u_y(0) = (G_{11} - i\mu_{N+1}\nu_{N+1}G_{12})B_{N+1}$

2. $0 = (G_{21} - i\mu_{N+1}\nu_{N+1}G_{22})B_{N+1}$

For a non-trivial solution ($B_{N+1} \neq 0$) the bracket in the second equation must vanish. This is our **Dispersion Equation**:

$$\boxed{D(\omega, c) = G_{21}(\omega, c) - i\mu_{N+1}(\omega)\nu_{N+1}(\omega, c)\, G_{22}(\omega, c) = 0}$$

The first equation then merely fixes the overall amplitude; $u_y(0)$ is arbitrary, as it must be for a homogeneous (source-free) eigenvalue problem. Note that D is only defined up to a nonzero multiplicative factor, since $\mathbf{V}$ could have been scaled arbitrarily. Only the location of its zeros carries physical meaning.

Summary and Numerical Solution for N Layers Over a Half-space

To find the Love wave modes of the N-layer system (the case $N = 5$ is worked out in detail below):

The dispersion equation is given as:

$$D(\omega, c) = G_{21}(\omega, c) - i\mu_{N+1}(\omega)\, \nu_{N+1}(\omega, c)\, G_{22}(\omega, c) = 0$$

1. **For a given real frequency** ω and a trial *complex* phase velocity c, evaluate the complex shear moduli $\mu_j(\omega)$ from the viscoelastic model, then the complex shear speeds $\beta_j(\omega) = \sqrt{\mu_j(\omega)/\rho_j}$ (branch $\operatorname{Re}\beta_j > 0$), then the vertical wavenumbers

$$\nu_j = \frac{\omega}{c}\sqrt{\left(\frac{c}{\beta_j}\right)^2 - 1}, \qquad j = 1, \dots, N+1.$$

   For $j \leq N$ the branch of the square root is immaterial, because $\mathbf{T}_j$ is an even function of ν_j. For the half-space the branch *must* be chosen so that $\operatorname{Im}(\nu_{N+1}) < 0$, i.e. flip the sign of the root if necessary.

2. **For each layer** $j = 1, \dots, N$, calculate its propagator matrix $\mathbf{T}_j$:

$$\mathbf{T}_j = \begin{bmatrix} \cos(\nu_j h_j) & -\dfrac{\sin(\nu_j h_j)}{\mu_j \nu_j} \\ \mu_j \nu_j \sin(\nu_j h_j) & \cos(\nu_j h_j) \end{bmatrix}$$

   Guard the $\nu_j \to 0$ limit by evaluating $\sin(\nu_j h_j)/\nu_j \to h_j$ directly (this is the removable singularity at $c = \beta_j$; the matrix itself is analytic there).

3. **Multiply the matrices, in order**, to get the global propagator:

$$\mathbf{G} = \mathbf{T}_1\mathbf{T}_2 \cdots \mathbf{T}_N$$

Verify $\det \mathbf{G} = 1$ to machine precision as a check.

4. **Evaluate the dispersion function:**

$$D(\omega, c) = G_{21} - i\mu_{N+1}\nu_{N+1}G_{22}$$

5. **Search for the roots** of $D(\omega, c) = 0$ in the complex c-plane (e.g. by a two-dimensional Newton or Muller iteration, or by the argument principle on a contour). A root is a trapped Love mode when

$$\min_{1\le j\le N} \mathrm{Re}\left(\beta_j\right) \;<\; \mathrm{Re}(c) \;<\; \mathrm{Re}\left(\beta_{N+1}\right),$$

that is, the phase velocity lies between the *slowest layer* and the half-space. (It is the minimum over all layers, not β_1, unless the layers happen to be ordered by increasing velocity. A trapped mode exists only if the half-space is faster than the slowest layer.) Roots with $\mathrm{Re}(c) > \mathrm{Re}(\beta_{N+1})$ are leaky modes, for which ν_{N+1} is no longer evanescent and the analysis above does not apply.

6. **Extract the observables.** The complex horizontal wavenumber is $k = \omega/c$. Writing $c = c_R + ic_I$,

$$k = \frac{\omega\, c_R}{|c|^2} - i\,\frac{\omega\, c_I}{|c|^2},$$

so the physical phase velocity and the spatial attenuation coefficient are

$$\boxed{c_{\mathrm{phase}}(\omega) = \frac{\omega}{\mathrm{Re}(k)} = \frac{|c|^2}{c_R}, \qquad \alpha(\omega) = -\,\mathrm{Im}(k) = \frac{\omega\, c_I}{|c|^2}, \qquad Q_L^{-1}(\omega) = \frac{2\alpha}{\mathrm{Re}(k)} = \frac{2c_I}{c_R}.}$$

Attenuation requires $c_I > 0$. **Note that $c_{\mathbf{phase}} \neq \mathrm{Re}(c)$**; the two agree only to first order in c_I/c_R, i.e. in the weak-attenuation limit.

7. Repeat over the frequency range to obtain the dispersion curve $c_{\mathrm{phase}}(\omega)$ and the attenuation curve $\alpha(\omega)$.

8. **Group velocity.** $D(\omega, k)$ is analytic in the complex variable k, so implicit differentiation along the mode branch $D(\omega, k(\omega)) = 0$ gives the *complex* derivative

$$\frac{dk}{d\omega} = -\,\frac{\partial D/\partial \omega}{\partial D/\partial k}.$$

Since the observable phase is governed by $\mathrm{Re}(k)$, the group velocity is

$$\boxed{U(\omega) = \frac{d\omega}{d\,\mathrm{Re}(k)} = \left[\mathrm{Re}\left(\frac{dk}{d\omega}\right)\right]^{-1}}$$

Note that one may *not* write $\partial D/\partial\,\mathrm{Re}(k)$: for an analytic D there is no separate derivative with respect to the real part. Take the full complex derivative first, then its real part, then invert. In the elastic limit $\mathrm{Im}(k) = 0$ and this reduces to the usual $U = d\omega/dk$.

The group velocity also sets the *level* of the modal attenuation. To first order in the layer dissipations, scaling all moduli $\mu_j \to s\mu_j$ maps $k(\omega) \mapsto k(\omega/\sqrt{s})$, whence $\sum_j \mu_j\,\partial k/\partial\mu_j = -\,\omega/(2U)$ and

$$Q_L^{-1} = \frac{c}{U}\sum_j w_j\,Q_j^{-1}, \qquad w_j \geq 0, \qquad \sum_j w_j = 1.$$

Because $U < c$ for these normally dispersive modes, Q_L generally falls *below* the smallest layer Q_j where the dispersion is steep: the mode quality factor is **not** bounded by the material quality factors, and treating $\min_j Q_j \leq Q_L \leq \max_j Q_j$ as a sanity check will produce false alarms.

This process systematically applies the **Thomson–Haskell method** to the Love wave problem, reducing the boundary value problem to a root-finding exercise.

A caveat on numerical conditioning. At large ωh_j with evanescent layers (ν_j strongly imaginary), $\cos(\nu_j h_j)$ and $\sin(\nu_j h_j)$ both grow like $e^{|\,\mathrm{Im}\,\nu_j|h_j}$, and the product $\mathbf{G}$ loses the significant digits that encode the decaying solution. This is the well-known high-frequency instability of the Thomson–Haskell scheme. For the 2×2 Love problem it is far milder than for Rayleigh waves, but at high frequency–thickness products it should be handled by normalizing $\mathbf{T}_j$ by $e^{|\,\mathrm{Im}\,\nu_j|h_j}$ at each step (the dispersion function's zeros are unaffected by a positive scalar factor), or by reformulating with reflection/transmission coefficients or the delta-matrix construction of Dunkin [4].

Calculating all elements of the global propagator matrix, G

Calculating all elements of the global propagator matrix $\mathbf{G}$ is the core computational step.

1. Mathematical Definition

For an N-layer system:

$$\mathbf{G} = \mathbf{T}_1 \mathbf{T}_2 \cdots \mathbf{T}_N$$

where each layer matrix is:

$$\mathbf{T}_j = \begin{pmatrix} \cos(\nu_j h_j) & -\dfrac{\sin(\nu_j h_j)}{\mu_j \nu_j} \\ \mu_j \nu_j \sin(\nu_j h_j) & \cos(\nu_j h_j) \end{pmatrix}$$

2. Step-by-Step Multiplication

Step 1: Define Layer Matrices Explicitly

For layer j, let:

$$\mathbf{T}_j = \begin{pmatrix} C_j & -S'_j \\ S_j & C_j \end{pmatrix}$$

where:

$$C_j = \cos(\nu_j h_j), \qquad S_j = \mu_j \nu_j \sin(\nu_j h_j), \qquad S'_j = \frac{\sin(\nu_j h_j)}{\mu_j \nu_j}$$

Useful identity:

$$S_j \cdot S'_j = \sin^2(\nu_j h_j), \qquad \text{so} \qquad \det \mathbf{T}_j = C_j^2 + S_j S'_j = 1.$$

Step 2: Multiply First Two Layers

Let $\mathbf{G}^{(2)} = \mathbf{T}_1 \mathbf{T}_2$.

$$\mathbf{G}^{(2)} = \begin{pmatrix} C_1 & -S'_1 \\ S_1 & C_1 \end{pmatrix} \begin{pmatrix} C_2 & -S'_2 \\ S_2 & C_2 \end{pmatrix}$$

Element-by-element calculation:

$$\begin{aligned} G_{11}^{(2)} &= C_1 C_2 - S'_1 S_2, \\ G_{12}^{(2)} &= -\left(C_1 S'_2 + S'_1 C_2\right), \\ G_{21}^{(2)} &= S_1 C_2 + C_1 S_2, \\ G_{22}^{(2)} &= C_1 C_2 - S_1 S'_2. \end{aligned}$$

Thus:

$$\mathbf{G}^{(2)} = \begin{pmatrix} C_1C_2 - S_1'S_2 & -(C_1S_2' + S_1'C_2) \\ S_1C_2 + C_1S_2 & C_1C_2 - S_1S_2' \end{pmatrix}$$

<u>**Step 3: General Recursive Formula**</u>

For $\mathbf{G}^{(k)} = \mathbf{G}^{(k-1)}\mathbf{T}_k$, with $\mathbf{G}^{(1)} = \mathbf{T}_1$:

$$\boxed{\begin{aligned} G_{11}^{(k)} &= G_{11}^{(k-1)}C_k + G_{12}^{(k-1)}S_k, \\ G_{12}^{(k)} &= -G_{11}^{(k-1)}S_k' + G_{12}^{(k-1)}C_k, \\ G_{21}^{(k)} &= G_{21}^{(k-1)}C_k + G_{22}^{(k-1)}S_k, \\ G_{22}^{(k)} &= -G_{21}^{(k-1)}S_k' + G_{22}^{(k-1)}C_k. \end{aligned}}$$

Applying this for $k = 2, 3, \ldots, N$ gives $\mathbf{G} = \mathbf{G}^{(N)}$, whose entries G_{21} and G_{22} are the only ones the dispersion function needs. Rows 1 and 2 of the recursion are decoupled, so if only $D(\omega, c)$ is required it suffices to propagate the single row vector (G_{21}, G_{22}), initialized at (S_1, C_1). The first row is needed only to reconstruct the eigenfunction $u_y(z)$.

<u>**Step 4: Worked case,** $N = 5$</u>

$$\mathbf{G} = \mathbf{G}^{(5)} = \big(\big((\mathbf{T}_1\mathbf{T}_2)\mathbf{T}_3\big)\mathbf{T}_4\big)\mathbf{T}_5,$$

and the dispersion function is

$$\boxed{D(\omega, c) = G_{21}^{(5)} - i\,\mu_6\,\nu_6\,G_{22}^{(5)}}$$

with μ_6, ν_6 the half-space modulus and vertical wavenumber (Im $\nu_6 < 0$).

<u>**Consistency check ($N = 1$).**</u> For a single layer over a half-space, $\mathbf{G} = \mathbf{T}_1$, so

$$D = \mu_1\nu_1 \sin(\nu_1 h_1) - i\mu_2\nu_2 \cos(\nu_1 h_1) = 0 \quad \Longrightarrow \quad \tan(\nu_1 h_1) = \frac{i\mu_2\nu_2}{\mu_1\nu_1}.$$

In the elastic case $\nu_2 = -i\gamma_2$ with $\gamma_2 = k\sqrt{1 - c^2/\beta_2^2}$, so $i\mu_2\nu_2 = \mu_2\gamma_2$ and

$$\tan\left(kh_1\sqrt{\tfrac{c^2}{\beta_1^2} - 1}\right) = \frac{\mu_2\sqrt{1 - c^2/\beta_2^2}}{\mu_1\sqrt{c^2/\beta_1^2 - 1}},$$

which is the classical Love-wave dispersion relation for a layer over a half-space. This recovers the textbook result exactly, and confirms both the signs in $\mathbf{T}_j$ and the branch Im $\nu_2 < 0$.

9 Viscoelastic Model

So far the details of the relaxation function are not defined. Therefore, the objective here is to find a **relaxation function with a frequency-independent $Q(\omega)$-value**. For the application in seismic modelling, it is important that the viscoelastic model can describe a frequency-independent $Q(\omega)$. We can construct viscoelastic models composed of two basic elements.

Generalized Maxwell-model

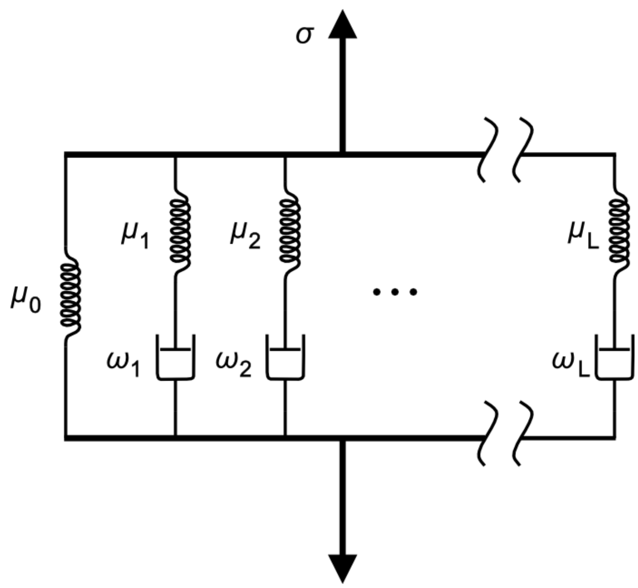

Figure 1: Generalized Maxwell Model

In GMB, we add multiple Maxwell models in parallel, which yields the Generalized Maxwell model or Generalized Maxwell body (GMB), also known as Maxwell-Wiechert model. By the superposition of multiple Maxwell models with different elastic moduli μ_l and viscosities η_l, we can achieve a constant Q-value over a given frequency range.

The **Hooke element** (spring), representing the linear elastic medium

$$\sigma_{Hooke} = \mu\epsilon$$
$$\text{or}$$
$$\epsilon_{Hooke} = \frac{\sigma}{\mu}$$

The **Newton element** (dashpot), representing the viscous damping part with the stress–strain-rate relation:

$$\sigma_{Newton} = \eta\dot{\epsilon}$$
$$\text{or}$$
$$\dot{\epsilon}_{Newton} = \frac{\sigma}{\eta}$$

where η denotes the viscosity of the medium.

Because we assemble the Maxwell SLS model (a lone spring μ_0 in parallel with one Maxwell body) with $L-1$ additional Maxwell bodies in parallel, and elements in parallel share the strain and add their stresses, we add the stresses in the frequency domain:

$$\tilde{\sigma}_{GMB} = \tilde{\sigma}_{SLSM} + \sum_{l=2}^{L} \tilde{\sigma}_{Maxwell,l}$$

Inserting the stresses

$$\begin{aligned} \tilde{\sigma}_{SLSM} &= \left(\mu_0 + \frac{i\mu_1\omega\eta_1}{i\omega\eta_1 + \mu_1}\right)\tilde{\epsilon} \\ \tilde{\sigma}_{Maxwell,l} &= \frac{i\mu_l\omega\eta_l}{i\omega\eta_l + \mu_l}\tilde{\epsilon} \end{aligned} \tag{8}$$

we have the <u>**frequency-domain stress-strain relation for the GMB:**</u>

$$\tilde{\sigma}_{GMB} = \left(\mu_0 + \frac{i\mu_1\omega\eta_1}{i\omega\eta_1 + \mu_1} + \sum_{l=2}^{L} \frac{i\mu_l\omega\eta_l}{i\omega\eta_l + \mu_l}\right)\tilde{\epsilon}$$

We can move the second term into the sum over the L Maxwell-models:

$$\tilde{\sigma}_{GMB} = \left(\mu_0 + \sum_{l=1}^{L} \frac{i\mu_l\omega\eta_l}{i\omega\eta_l + \mu_l}\right)\tilde{\epsilon}$$

Dividing numerator and denominator of each term by η_l and introducing the **relaxation frequencies:**

$$\omega_l := \frac{\mu_l}{\eta_l}$$

leads to

$$\tilde{\sigma}_{GMB} = \left(\mu_0 + \sum_{l=1}^{L} \frac{i\mu_l\omega}{i\omega + \frac{\mu_l}{\eta_l}}\right)\tilde{\epsilon} = \left(\mu_0 + \sum_{l=1}^{L} \frac{i\mu_l\omega}{i\omega + \omega_l}\right)\tilde{\epsilon}$$

We now simplify the complex modulus

$$\boxed{\tilde{\mu}_{GMB} = \mu_0 + \sum_{l=1}^{L} \frac{i\mu_l\omega}{i\omega + \omega_l}}$$

First we estimate the relaxed shear modulus:

$$\tilde{\mu}_{GMB,R} = \lim_{\omega\to 0} \tilde{\mu}_{GMB} = \mu_0$$

and the unrelaxed shear modulus. Factor $i\omega$ out of each denominator:

$$\tilde{\mu}_{GMB}(\omega) = \mu_0 + \sum_{l=1}^{L} \frac{i\mu_l\omega}{i\omega\left(1 + \frac{\omega_l}{i\omega}\right)} = \mu_0 + \sum_{l=1}^{L} \frac{\mu_l}{1 - i\frac{\omega_l}{\omega}}$$

using $1/i = -i$. As $\omega \to \infty$, $\omega_l/\omega \to 0$, so each term tends to μ_l:

$$\tilde{\mu}_{GMB,U} = \lim_{\omega\to\infty} \tilde{\mu}_{GMB}(\omega) = \mu_0 + \sum_{l=1}^{L} \mu_l$$

With the **modulus defect** or **relaxation of modulus**

$$\delta\mu = \tilde{\mu}_{GMB,U} - \tilde{\mu}_{GMB,R} = \left(\mu_0 + \sum_{l=1}^{L} \mu_l\right) - \mu_0 = \sum_{l=1}^{L} \mu_l$$

For individual mechanisms:

$$\delta\mu_l = \mu_l$$

since each Maxwell body contributes μ_l to the total modulus defect.

Normalization with weights a_l

Write each branch defect as a fraction of the total defect,

$$\delta\mu_l = a_l\, \delta\mu$$

where the weights a_l are, by construction, normalized:

$$\sum_{l=1}^{L} a_l = 1$$

Since $\delta\mu_l = \mu_l$, this gives $\mu_l = a_l\, \delta\mu$. Note that this normalization is a choice: it fixes $\delta\mu = \mu_u - \mu_0$ once the relaxed and unrelaxed moduli are specified, so the constant-Q fit below determines the a_l subject to $\sum_l a_l = 1$, with the ω_l prescribed.

So, each μ_l is expressed as a fraction of the total modulus defect:

$$\mu_l = a_l \delta\mu$$

Verification:

$$\sum_{l=1}^{L} \mu_l = \sum_{l=1}^{L} a_l \delta\mu = \delta\mu \sum_{l=1}^{L} a_l = \delta\mu \cdot 1 = \delta\mu$$

which matches our earlier result.

Substitute $\mu_l = a_l \delta\mu$ into the original expression:

$$\tilde{\mu}_{GMB}(\omega) = \mu_0 + \sum_{l=1}^{L} \frac{i(a_l \delta\mu)\omega}{i\omega + \omega_l}$$

$$\boxed{\tilde{\mu}_{GMB}(\omega) = \mu_0 + \delta\mu \sum_{l=1}^{L} \frac{i a_l \omega}{i\omega + \omega_l}} \tag{9}$$

where μ_0 denotes the **relaxed shear modulus**, $\delta\mu$ the **modulus defect**, L the number of Maxwell bodies, a_l, ω_l the **weighting coefficients** and **relaxation frequencies** of the l-th Maxwell body to achieve a constant Q-spectrum, while ω is the circular frequency within the frequency range of the source wavelet.

Final Note:

1. Low-frequency limit: All Maxwell bodies are relaxed → only μ_0 remains
2. High-frequency limit: All Maxwell bodies are stiff → each contributes μ_l
3. Modulus defect: Difference between high and low frequency limits $= \sum \mu_l$
4. Weight normalization: Distribute total defect among mechanisms with weights a_l

Quality Factor of the GMB

The whole point of the GMB is to realize a prescribed, nearly frequency-independent Q, so we must exhibit $Q(\omega)$ explicitly. Rationalize each term of (9) by multiplying numerator and denominator by the conjugate $(\omega_l - i\omega)$:

$$\frac{ia_l\omega}{i\omega + \omega_l} = \frac{ia_l\omega\,(\omega_l - i\omega)}{(\omega_l + i\omega)(\omega_l - i\omega)} = \frac{a_l\omega^2 + i\,a_l\omega\omega_l}{\omega^2 + \omega_l^2}.$$

Hence the real and imaginary parts of the complex shear modulus are

$$\boxed{\mathrm{Re}\;\tilde{\mu}_{GMB}(\omega) = \mu_0 + \delta\mu \sum_{l=1}^{L} \frac{a_l\,\omega^2}{\omega^2 + \omega_l^2}, \qquad \mathrm{Im}\;\tilde{\mu}_{GMB}(\omega) = \delta\mu \sum_{l=1}^{L} \frac{a_l\,\omega\,\omega_l}{\omega^2 + \omega_l^2}} \tag{10}$$

If $\delta\mu > 0$, $\omega_l > 0$ and every $a_l \geq 0$, then $\mathrm{Im}\;\tilde{\mu} > 0$ for all $\omega > 0$ and the model is dissipative, as thermodynamics requires. This is a *condition on the weights*, not an automatic property of the form: an unconstrained fit can return negative a_l, and passivity must then be checked or enforced (see below). With the convention fixed at the outset,

$$\boxed{Q^{-1}(\omega) = \frac{\mathrm{Im}\;\tilde{\mu}_{GMB}(\omega)}{\mathrm{Re}\;\tilde{\mu}_{GMB}(\omega)} = \frac{\delta\mu \displaystyle\sum_{l=1}^{L} \frac{a_l\,\omega\,\omega_l}{\omega^2 + \omega_l^2}}{\mu_0 + \delta\mu \displaystyle\sum_{l=1}^{L} \frac{a_l\,\omega^2}{\omega^2 + \omega_l^2}}} \tag{11}$$

A single Maxwell body ($L = 1$) gives a Debye peak: Q^{-1} is maximal near $\omega = \omega_1$ and falls off as $\omega^{\pm 1}$ away from it. It cannot produce a constant Q. The construction of Emmerich and Korn [5] is to place the L relaxation frequencies ω_l logarithmically equispaced across the frequency band of interest $[\omega_{\min}, \omega_{\max}]$ and then choose the weights a_l so that (11) matches a target constant Q_0.

It is worth being careful here, because the fitting problem is often described as nonlinear and then linearized, which is both unnecessary and inaccurate. Introduce the **dimensionless weights**

$$\hat{a}_l := \frac{\delta\mu}{\mu_u}\, a_l, \qquad \text{so that} \qquad \frac{\mu_0}{\mu_u} = 1 - \sum_{l=1}^{L} \hat{a}_l.$$

Demanding $Q^{-1}(\omega) = Q_0^{-1}$ in (11) means Im $\tilde{\mu} = Q_0^{-1}$ Re $\tilde{\mu}$. Dividing through by μ_u,

$$\sum_{l=1}^{L} \hat{a}_l \,\frac{\omega\,\omega_l}{\omega^2 + \omega_l^2} = Q_0^{-1}\left[1 - \sum_{l=1}^{L} \hat{a}_l \,\frac{\omega_l^2}{\omega^2 + \omega_l^2}\right],$$

where the bracket has been simplified using $1 - \hat{a}_l + \hat{a}_l\,\omega^2/(\omega^2 + \omega_l^2) = 1 - \hat{a}_l\,\omega_l^2/(\omega^2 + \omega_l^2)$. Collecting the $\hat{a}_l$ on one side shows that the condition is **exactly linear** in the weights, with no approximation whatsoever:

$$\boxed{\sum_{l=1}^{L} \frac{\omega_i\,\omega_l + Q_0^{-1}\,\omega_l^2}{\omega_i^2 + \omega_l^2}\;\hat{a}_l \;=\; Q_0^{-1}} \tag{12}$$

imposed at a set of collocation frequencies ω_i (in practice a few per mechanism, logarithmically spaced across the band) and solved in the least-squares sense. The relaxed and unrelaxed moduli then follow from $\delta\mu/\mu_u = \sum_l \hat{a}_l$ and $a_l = \hat{a}_l / \sum_m \hat{a}_m$, which automatically satisfies the normalization $\sum_l a_l = 1$.

Dropping the $Q_0^{-1}\omega_l^2$ term in (12) – which is what the "linearize by setting the denominator $\approx \mu_u$" shortcut amounts to – does not merely degrade the fit, it *biases* it: the recovered Q is systematically too low by roughly $\delta\mu/\mu_u$, of order 10% for $Q_0 = 50$. With (12) solved exactly, three mechanisms per decade hold Q to about 2% and five per decade to well under 1% across three decades; one per decade is not enough (errors above 20%) [5, 7].

Passivity. The unconstrained least-squares solution of (12) may contain small negative $\hat{a}_l$, which violates the sufficient condition for Im $\tilde{\mu} > 0$ established above. In practice Im $\tilde{\mu}$ usually remains positive anyway, because one negative weight is outweighed by its neighbours, but this must be verified rather than assumed. If per-mechanism passivity is required, solve (12) by non-negative least squares; the cost is a modest loss of accuracy (roughly 2% instead of 0.4% at five mechanisms per decade).

Pinning the model to a measured velocity. Since Re $\tilde{\mu}$ varies with frequency, "the" shear velocity of a layer is meaningless until a reference frequency ω_{ref} is named. In practice one specifies the measured phase velocity β_j^{ref} at ω_{ref} and rescales the moduli so that

$$\mathrm{Re}[\beta_j(\omega_{\mathrm{ref}})] = \beta_j^{\mathrm{ref}}, \qquad \beta_j(\omega) = \sqrt{\tilde{\mu}_j(\omega)/\rho_j}, \qquad \mathrm{Re}\,\beta_j > 0.$$

(Numerically it is often more convenient to enforce Re $\tilde{\mu}_j(\omega_{\text{ref}}) = \rho_j\left(\beta_j^{\text{ref}}\right)^2$ instead; the two prescriptions agree to $O(Q^{-2})$.) Omitting this step leaves the overall velocity level of the model undetermined, and the computed dispersion curves are then offset by a constant.

Relaxation Function Derivation: Transformation of Complex Modulus to Time Domain

Deriving the Relaxation Function

We need to transform the complex modulus above to time-domain by inverse Fourier transform leading to the relaxation function.

Given the complex modulus in the frequency domain:

$$\tilde{\mu}_{GMB}(\omega) = \mu_0 + \delta\mu \sum_{l=1}^{L} \frac{ia_l\omega}{i\omega + \omega_l} \equiv \mu_R + \delta\mu \sum_{l=1}^{L} \frac{ia_l\omega}{i\omega + \omega_l}$$

where from here on we write $\mu_R \equiv \mu_0$ for the relaxed modulus. (Do not confuse μ_R with μ', the real part of $\tilde{\mu}$; they coincide only as $\omega \to 0$.)
We want to transform this to the **time-domain relaxation modulus** $\Psi(t)$.

Relationship between complex modulus and relaxation modulus

In linear viscoelasticity, the complex modulus $\tilde{\mu}(\omega)$ is related to the relaxation modulus $\Psi(t)$ via:

$$\tilde{\mu}(\omega) = i\omega\, \mathcal{F}[\Psi(t)](\omega)$$

where $\mathcal{F}[\Psi(t)](\omega) = \int_0^\infty \Psi(t)e^{-i\omega t}dt$ is the Fourier transform (for causal $\Psi(t)$).

$$\tilde{\mu}(\omega) = i\omega \int_0^\infty \Psi(t)e^{-i\omega t}dt$$

$$\tilde{\mu}(\omega) = i\omega\, \widehat{\Psi}(\omega)$$

This means $\tilde{\mu}(\omega)$ is the Fourier transform of the distributional derivative of $\Psi(t)$ (distributional because Ψ jumps at $t = 0$, contributing $\mu_u\,\delta(t)$), or equivalently:

$$\frac{\tilde{\mu}(\omega)}{i\omega} = \int_0^\infty \Psi(t)e^{-i\omega t}dt$$

Thus, $\Psi(t)$ is the inverse Fourier transform of $\tilde{\mu}(\omega)/(i\omega)$.

So to get $\Psi(t)$, we take the inverse Fourier transform:

$$\Psi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{\mu}(\omega)}{i\omega} e^{i\omega t}\, d\omega.$$

Because $\tilde{\mu}(0) = \mu_0 \neq 0$, the integrand has a simple pole at $\omega = 0$ and the integral must be read as a principal value plus a half-residue. It is cleaner to route the inversion through the Laplace transform, which we do next.

Rewrite the Complex Modulus

Divide the given expression by $i\omega$:

$$\frac{\tilde{\mu}(\omega)}{i\omega} = \frac{\mu_0}{i\omega} + \delta\mu \sum_{l=1}^{L} \frac{a_l}{i\omega + \omega_l}$$

Switch to Laplace domain

Using the substitution $s = i\omega$ (one-sided Fourier transform)

Relaxation modulus $\Psi(t)$ has Laplace transform $\overline{\Psi}(s)$ with:

$$\tilde{\mu}(\omega) = s\,\overline{\Psi}(s)\big|_{s=i\omega}.$$

So:

$$\overline{\Psi}(s) = \frac{\tilde{\mu}(s)}{s} = \frac{\mu_0}{s} + \delta\mu \sum_{l=1}^{L} \frac{a_l}{s + \omega_l}$$

where $\overline{\Psi}(s)$ is the Laplace transform of $\Psi(t)$.

Inverse Laplace transform

Taking the inverse Laplace transform term by term:

We know:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad t \geq 0$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+\omega_l}\right\} = e^{-\omega_l t}$$

Final time-domain expression

Therefore, the relaxation modulus in the time domain is:

$$\boxed{\Psi(t) = \mu_0 + \delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t}} \tag{13}$$

for $t \geq 0$.

$$\boxed{\Psi(t) = \left\{\mu_0 + \delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t}\right\} \cdot H(t)} \tag{14}$$

where $H(t)$ is the Heaviside step function.

This is the **stress relaxation function** corresponding to the given **complex modulus** (9).

Using the definition of the modulus defect as the difference between the unrelaxed μ_u and relaxed shear modulus μ_0:

$$\delta\mu = \mu_u - \mu_0$$

we can replace the relaxed by the unrelaxed modulus [5, 7]:

$$\boxed{\Psi(t) = \left\{\mu_u - \delta\mu \sum_{l=1}^{L} a_l \left(1 - e^{-\omega_l t}\right)\right\} \cdot H(t)} \tag{15}$$

So, we are going to use this function in our dispersion calculation above.

Transforming the Relaxation Function (15) to Frequency Domain

Given:

$$\Psi(t) = \left\{\mu_u - \delta\mu \sum_{l=1}^{L} a_l \left(1 - e^{-\omega_l t}\right)\right\} \cdot H(t)$$

where $H(t)$ is the Heaviside step function.

Rewrite:

$$\Psi(t) = \mu_u H(t) - \delta\mu \sum_{l=1}^{L} a_l \left[1 - e^{-\omega_l t}\right] H(t)$$

$$\Psi(t) = \mu_u H(t) - \delta\mu \sum_{l=1}^{L} a_l H(t) + \delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t} H(t)$$

Note: $\sum_{l=1}^{L} a_l$ is just a constant.

Let $A = \sum_{l=1}^{L} a_l$.

Then:

$$\Psi(t) = \mu_u H(t) - \delta\mu\, A\, H(t) + \delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t} H(t)$$

$$\Psi(t) = \left[\mu_u - \delta\mu A\right] H(t) + \delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t} H(t)$$

<u>Fourier transform</u>

We use the Fourier transform definition:

$$\mathcal{F}\{f(t)\}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

We know:

$$\mathcal{F}\{H(t)\}(\omega) = \pi\delta(\omega) + \frac{1}{i\omega}$$

(in the sense of distributions; the $1/(i\omega)$ is to be read as a Cauchy principal value, which is the content of the Sokhotski–Plemelj formula).

Also:

$$\mathcal{F}\{e^{-\alpha t} H(t)\} = \frac{1}{\alpha + i\omega}, \quad \alpha > 0.$$

<u>Transform of first term</u>

First term: $(\mu_u - \delta\mu A) H(t)$

$$\mathcal{F}\left\{(\mu_u - \delta\mu A)H(t)\right\} = (\mu_u - \delta\mu A)\left[\pi\delta(\omega) + \frac{1}{i\omega}\right]$$

Transform of second term

Second term: $\delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t} H(t)$

$$\mathcal{F}\left\{\delta\mu\, a_l e^{-\omega_l t} H(t)\right\} = \delta\mu\, a_l \cdot \frac{1}{\omega_l + i\omega}$$

So:

$$\mathcal{F}\left\{\delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t} H(t)\right\} = \delta\mu \sum_{l=1}^{L} \frac{a_l}{\omega_l + i\omega}$$

Combine

$$\boxed{\widetilde{\Psi}(\omega) = (\mu_u - \delta\mu A)\left[\pi\delta(\omega) + \frac{1}{i\omega}\right] + \delta\mu \sum_{l=1}^{L} \frac{a_l}{\omega_l + i\omega}}$$

Simplify the constant A

Recall:

$$A = \sum_{l=1}^{L} a_l$$

We can combine the $1/(i\omega)$ term with the sum over $a_l/(\omega_l + i\omega)$ if desired, but in rheology and viscoelasticity it is customary to keep it as a sum of Debye terms plus a singular term at $\omega = 0$.

Let's check: The $\frac{1}{i\omega}$ term coefficient is $\mu_u - \delta\mu A$.
Under the normalization $\sum_l a_l = 1$ adopted above, $A = 1$ and this coefficient is exactly $\mu_u - \delta\mu = \mu_0$. More generally, $\Psi(t \to \infty) = \mu_u - \delta\mu A = \mu_R$ (relaxed modulus) and $\Psi(0^+) = \mu_u$ (unrelaxed modulus). Indeed, at $t = 0^+$, $e^{-\omega_l t} = 1$, so

$$\Psi(0^+) = \mu_u - \delta\mu \sum_{l=1}^{L} a_l(1 - 1) = \mu_u.$$

At $t \to \infty$, $e^{-\omega_l t} \to 0$, so

$$\Psi(\infty) = \mu_u - \delta\mu \sum_{l=1}^{L} a_l = \mu_u - \delta\mu A.$$

So indeed, $\mu_R = \mu_u - \delta\mu A$.

Thus:

$$\boxed{\widetilde{\Psi}(\omega) = \mu_R \left[\pi\delta(\omega) + \frac{1}{i\omega}\right] + \delta\mu \sum_{l=1}^{L} \frac{a_l}{\omega_l + i\omega}}$$

This is the relaxation function (15) in the frequency domain.

Interpret in complex modulus $G^*(\omega)$

In rheology, the complex modulus $G^*(\omega) = i\omega\widetilde{\Psi}(\omega)$ for a stress relaxation modulus $\Psi(t)$.

Let's compute $G^*(\omega)$:

$$G^*(\omega) = i\omega\widetilde{\Psi}(\omega) = i\omega\,\mathcal{F}[\Psi(t)]$$

$$= i\omega \left[\mu_R \left(\pi\delta(\omega) + \frac{1}{i\omega}\right) + \delta\mu \sum_{l=1}^{L} \frac{a_l}{\omega_l + i\omega}\right]$$

The term $i\omega \cdot \mu_R \pi\delta(\omega) = i\pi\mu_R\omega\delta(\omega) = 0$ because $\omega\delta(\omega) = 0$.

The term $i\omega \cdot \frac{\mu_R}{i\omega} = \mu_R$.

So:

$$G^*(\omega) = \mu_R + \delta\mu \sum_{l=1}^{L} \frac{i\omega a_l}{\omega_l + i\omega}$$

This is a standard generalized Maxwell model form:

$$G^*(\omega) = \mu_R + \sum_{l=1}^{L} \frac{i\omega\delta\mu a_l}{\omega_l + i\omega}$$

with relaxation strengths $\delta\mu a_l$ for mode l.

Relaxation modulus $\Psi(t)$ in frequency domain

$$\boxed{\widetilde{\Psi}(\omega) = \mu_R\left[\pi\delta(\omega) + \frac{1}{i\omega}\right] + \delta\mu\sum_{l=1}^{L}\frac{a_l}{\omega_l + i\omega}}$$

where $\mu_R = \mu_u - \delta\mu\sum_{l=1}^{L} a_l$.

In summary, $G^*(\omega) = i\omega\widetilde{\Psi}(\omega)$ returns the complex modulus of the Generalized Maxwell body, closing the loop with (9). (Ψ already carries the factor $H(t)$, so no additional $H(t)$ is written.)

GMB stress-strain relation

We now insert the relaxation function into the viscoelastic stress-strain relation. Integrating $\sigma = \Psi * \dot{\epsilon}$ by parts, and using causality, gives the equivalent form

$$\sigma(t) = \int_{-\infty}^{t^+} \dot{\Psi}(t-t')\,\epsilon(t')\,dt' = \dot{\Psi} * \epsilon$$

where the upper limit is written t^+ because $\dot{\Psi}$ contains a Dirac delta at zero argument, and the whole of that delta must be captured.

We therefore need the first time derivative of $\Psi(t)$:

$$\boxed{\Psi(t) = \left\{\mu_u - \delta\mu\sum_{l=1}^{L} a_l\left(1 - e^{-\omega_l t}\right)\right\}\cdot H(t)} \tag{16}$$

$$\boxed{\dot{\Psi}(t) = -\delta\mu\sum_{l=1}^{L} a_l\omega_l e^{-\omega_l t}\cdot H(t) + \left\{\mu_u - \delta\mu\sum_{l=1}^{L} a_l\left(1 - e^{-\omega_l t}\right)\right\}\cdot\delta(t)}$$

This is the **time derivative of the relaxation function** $\Psi(t)$.

But at $t = 0$, $e^{-\omega_l t} = 1$ (since $\delta(t)$ is supported only at $t = 0$), so:

$$\mu_u - \delta\mu\sum_{l=1}^{L} a_l(1-1) = \mu_u.$$

Thus the given $\delta(t)$ coefficient simplifies to μ_u, matching our result.

Final derivative

$$\boxed{\dot{\Psi}(t) = -\delta\mu \sum_{l=1}^{L} a_l \omega_l e^{-\omega_l t} H(t) + \mu_u\, \delta(t)}$$

with $\delta\mu = \mu_u - \mu_0$.

Insert into stress-strain relation

$$\sigma(t) = \int_{-\infty}^{t^+} \dot{\Psi}(t-t')\, \epsilon(t')\, dt'$$

The $\mu_u\, \delta(t-t')$ term sifts out $\mu_u\, \epsilon(t)$, leaving

$$= \mu_u \epsilon(t) - \delta\mu \sum_{l=1}^{L} a_l \omega_l \int_{-\infty}^{t} e^{-\omega_l (t-t')} H(t-t') \epsilon(t')\, dt'$$

Since $H(t-t') = 1$ for $t' \le t$, we can write:

$$\boxed{\sigma(t) = \mu_u \epsilon(t) - \delta\mu \sum_{l=1}^{L} a_l \omega_l \int_{-\infty}^{t} e^{-\omega_l (t-t')} \epsilon(t')\, dt'}$$

This is the **viscoelastic stress-strain relation** for the GMB model. The first term is the instantaneous (unrelaxed) elastic response; the second is the memory integral that removes, with a lag $1/\omega_l$ per mechanism, the excess stiffness $\delta\mu$.

Two limiting checks.

- **Static strain.** Set $\epsilon(t) = \epsilon_0$ for all t. Then $\int_{-\infty}^{t} e^{-\omega_l (t-t')}\, dt' = 1/\omega_l$, so

  $$\sigma = \mu_u \epsilon_0 - \delta\mu\, \epsilon_0 \sum_{l=1}^{L} a_l = (\mu_u - \delta\mu)\, \epsilon_0 = \mu_0\, \epsilon_0,$$

  the relaxed response, as it must be.

- **Instantaneous loading.** A strain switched on at $t = 0$ gives $\sigma(0^+) = \mu_u\, \epsilon(0^+)$, the unrelaxed response, because the memory integral has had no time to accumulate.

Transforming back with $\mathcal{F}[\,\epsilon(t)\,] = \tilde{\epsilon}$ and $\mathcal{F}\left[\int_{-\infty}^{t} e^{-\omega_l (t-t')} \epsilon(t') dt'\right] = \tilde{\epsilon}/(\omega_l + i\omega)$ recovers

$$\tilde{\sigma} = \left[\mu_u - \delta\mu \sum_{l=1}^{L} \frac{a_l\, \omega_l}{\omega_l + i\omega}\right] \tilde{\epsilon} = \left[\mu_0 + \delta\mu \sum_{l=1}^{L} \frac{i a_l\, \omega}{i\omega + \omega_l}\right] \tilde{\epsilon},$$

where the last step uses $\mu_u = \mu_0 + \delta\mu \sum_l a_l$ and $1 - \omega_l/(\omega_l + i\omega) = i\omega/(\omega_l + i\omega)$. This is exactly (9), so the time-domain and frequency-domain descriptions are consistent, and the loop closes.

10 Numerical Results

We now exercise the method on a concrete model. All results in this section were produced by the accompanying Python implementation of the scheme of the preceding sections (propagator assembly, constant-Q GMB calibration, and complex-root search), which passes the built-in consistency checks on every run: $\det \mathbf{G} = 1$ to machine precision, the assembled single-layer D agrees with the closed form $\mu_1\nu_1 \sin(\nu_1 h_1) - i\mu_2\nu_2 \cos(\nu_1 h_1)$ exactly, the near-elastic limit ($Q = 10^5$) reproduces the classical elastic Love root to a relative difference of 10^{-10}, and the modal eigenfunctions satisfy the free-surface condition $\sigma_{yz}(0) = 0$ to a scale-free residual of order 10^{-16} while decaying monotonically into the half-space.

Model 1

Model 1 is the near-surface profile of Table 4 of Yuan et al. [9]: three soft layers of equal thickness over a stiffer half-space (Table 1).

Layer	h (m)	ρ (kg/m^3)	β^{ref} (m/s)	Q
1	5	2000	180	18
2	5	2000	300	30
3	5	2000	420	42
half-space	∞	2000	500	50

Table 1: Model 1 [9]. β^{ref} is the shear phase velocity pinned at the reference frequency $f_{\mathrm{ref}} = 1$ Hz.

Constant-Q calibration

Each distinct Q is realized by a GMB with 15 Maxwell mechanisms logarithmically spaced over 0.025–160 Hz (four per decade, fitted well beyond the swept band), with weights obtained from the exactly linear collocation system (12) in the least-squares sense. The realized $Q(\omega)$ deviates from its target by about 0.5% across the entire swept band for every layer. The unconstrained fit returns some small negative weights $\hat{a}_l$; positivity of Im $\tilde{\mu}(\omega)$ (passivity) was verified numerically across the band, as required by the discussion following (10). The moduli are pinned at $f_{\mathrm{ref}} = 1$ Hz via Re $\tilde{\mu}_j(\omega_{\mathrm{ref}}) = \rho_j \left(\beta_j^{\mathrm{ref}}\right)^2$.

Dispersion and attenuation

The dispersion function $D(\omega, c)$ was swept over 0.1–80 Hz at 200 frequencies; at each frequency the complex roots were located by a real-axis scan seeding a complex Newton iteration, with roots carried forward as continuation seeds (Fig. 2, Fig. 3).

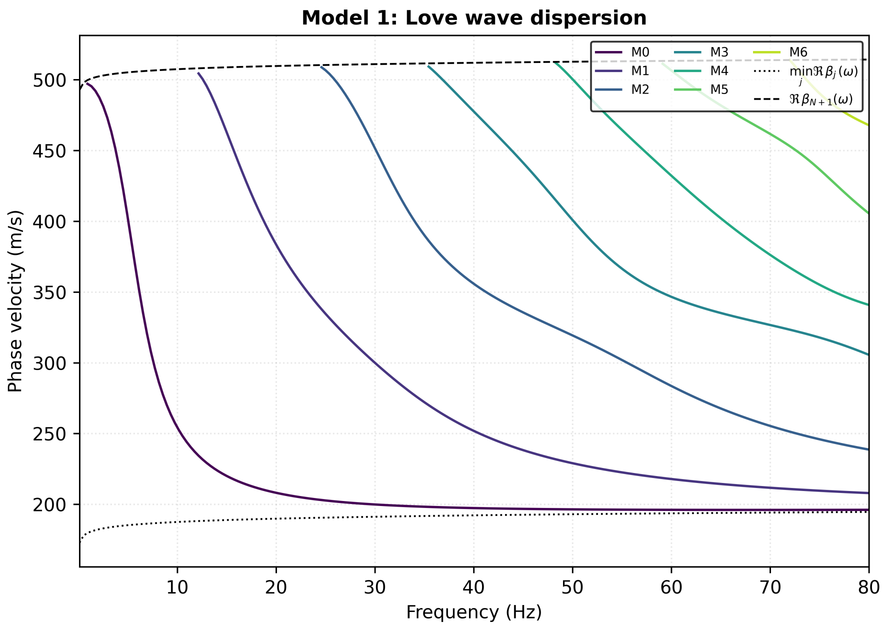

Figure 2: Model 1 Love-wave phase velocities $c_{\text{phase}} = |c|^2/\operatorname{Re}(c)$. The dotted and dashed curves are the frequency-dependent bounds $\min_j \operatorname{Re}\,\beta_j(\omega)$ and $\operatorname{Re}\,\beta_{N+1}(\omega)$ of the trapped-mode window; note that both drift upward with frequency because of the velocity dispersion that necessarily accompanies constant Q.

The fundamental mode enters at the half-space velocity and descends toward the velocity of the slowest layer: at 2 Hz the single trapped mode has $c_{\text{phase}} = 488.7$ m/s, and by 80 Hz the fundamental has flattened onto $c_{\text{phase}} = 196.1$ m/s. Six overtones enter successively across the band, each at its cutoff at the half-space velocity, so that seven trapped modes coexist at 80 Hz. Two consistency checks are worth making explicit. First, every root lies strictly inside the trapped-mode window at its own frequency. Second, the high-frequency limit of the fundamental, 196 m/s, sits a few m/s *above* $\operatorname{Re}\,\beta_1(2\pi \cdot 80\,\text{Hz}) \approx 194$ m/s – not above the 1-Hz table value of 180 m/s – because between 1 and 80 Hz the constant-Q material disperses by $\Delta\beta/\beta \approx (\pi Q_1)^{-1} \ln(80) \approx 8\%$. A code that ignored this dispersion would place the asymptote near 180 m/s and be wrong by the same 8%.

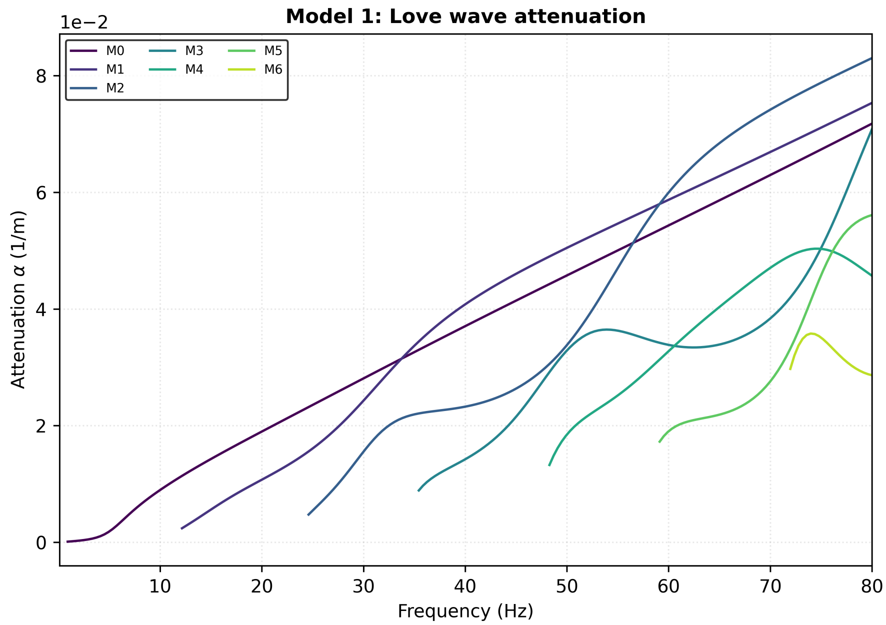

Figure 3: Model 1 spatial attenuation $\alpha = -\operatorname{Im}(k) = \omega \operatorname{Im}(c)/|c|^2$ from the exact complex roots of D.

The attenuation of each mode grows essentially linearly with frequency, as it must for frequency-independent material Q, with each overtone entering at low α near its cutoff (where the mode still lives mostly in the weakly dissipative half-space) and steepening as it concentrates into the shallow low-Q layers. The fundamental reaches $\alpha \approx 7 \times 10^{-2}\ \mathrm{m}^{-1}$ at 80 Hz.

The modal quality factor $Q_L = \operatorname{Re}(c)/(2 \operatorname{Im} c)$ makes the point of the group-velocity remark of the previous section concrete: the fundamental has $Q_L = 45.3$ at 2 Hz (the mode samples mostly the $Q = 50$ half-space) and $Q_L = 17.9$ at 80 Hz (the mode is confined to the $Q_1 = 18$ layer with $c/U \to 1$), while the strongly dispersive overtones dip to $Q_L \approx 11$–12 — *below* the smallest material Q of 18, exactly as the factor $c/U > 1$ in $Q_L^{-1} = (c/U) \sum_j w_j Q_j^{-1}$ predicts.

The simplified estimate, and why it fails

A common shortcut estimates the attenuation as $\alpha \approx \omega/(2c\,Q_{\text{eff}})$ with Q_{eff} a thickness-weighted average of the layer Q's ($Q_{\text{eff}} = 30$ for Model 1), shown in Fig. 4.

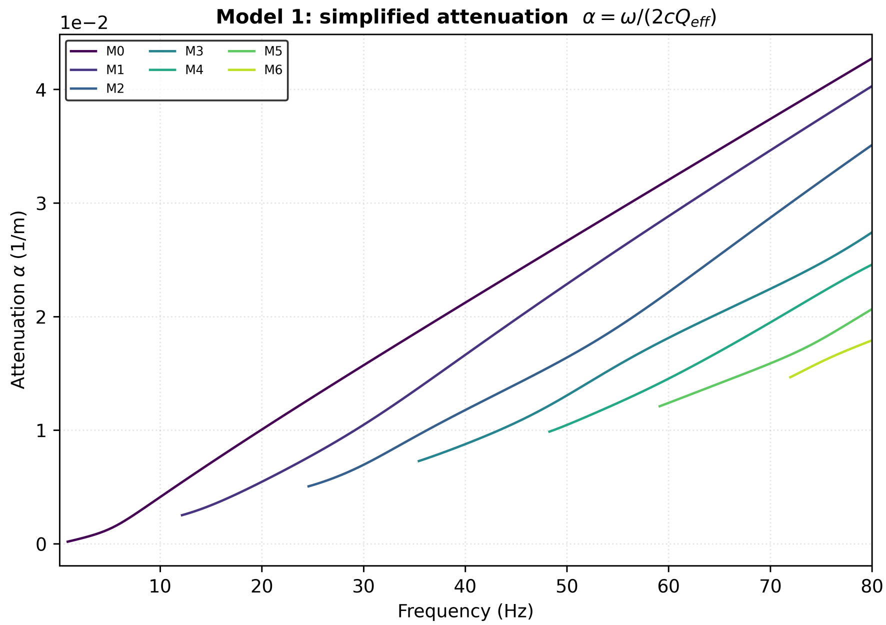

Figure 4: The simplified estimate $\alpha = \omega/(2c\,Q_{\text{eff}})$ evaluated along the same dispersion branches.

Comparison with Fig. 3 shows the estimate is wrong in both directions: at low frequency it *overestimates* α (the mode actually samples the higher-Q half-space, not the average), and at high frequency it *underestimates* the fundamental's α by roughly 40% (the mode feels only the $Q = 18$ layer, and the c/U factor is absent from the estimate altogether). The mode shape, which the simplified formula discards, is precisely what the propagator-matrix eigenvalue problem supplies.

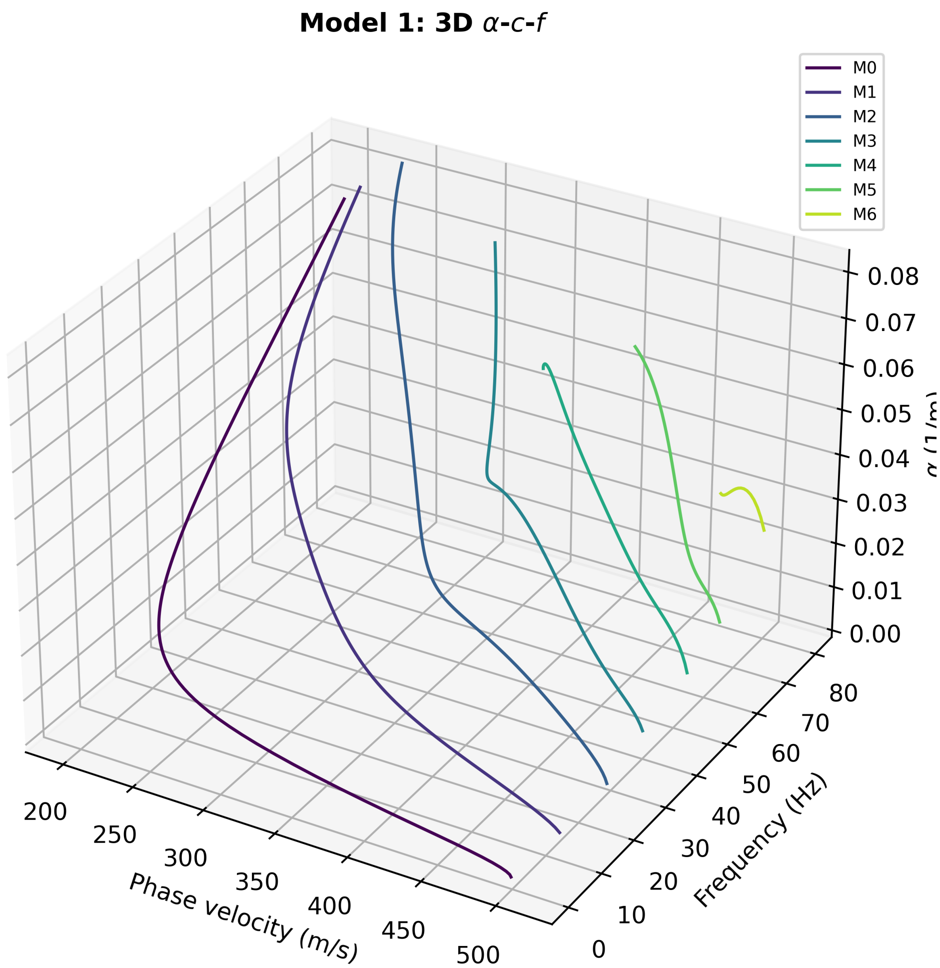

Figure 5: Model 1 modal trajectories in the frequency–phase velocity–attenuation domain.

11 Summary

Viscoelasticity introduces complex-valued moduli and hence complex wavenumbers, producing frequency-dependent phase velocities and attenuation that are absent in a purely elastic medium. The chain of reasoning assembled above is:

1. The equation of motion is indifferent to the constitutive law, so only the stress–strain relation changes: the elastic product $\sigma = 2\mu\varepsilon$ becomes the hereditary convolution $\sigma = \Psi_\mu * \dot{\varepsilon}$, which is a multiplication $\tilde{\sigma} = \tilde{\mu}(\omega)\tilde{\varepsilon}$ in the frequency domain. This is the correspondence principle.

2. Restricting to SH motion collapses the problem to the scalar field $u_y(x, z)$ and the two-component state vector $\mathbf{f} = (u_y, \sigma_{yz})^\mathsf{T}$, whose components are exactly the quantities

continuous across a welded interface.

3. Within a homogeneous layer, $\mathbf{f}$ obeys a linear first-order system whose solution operator over a thickness h_j is the unimodular propagator $\mathbf{T}_j$. Chaining the layers gives the global propagator $\mathbf{G} = \prod_j \mathbf{T}_j$.

4. A free surface at the top ($\sigma_{yz} = 0$) and a radiation condition at the bottom ($\mathrm{Im}\ \nu_{N+1} < 0$) turn the propagation relation into the scalar dispersion function $D(\omega, c) = G_{21} - i\mu_{N+1}\nu_{N+1}G_{22}$.

5. In the elastic case D is real on the physical interval and its roots are the familiar Love modes. In the viscoelastic case the moduli are complex, so the roots c migrate off the real axis; their real and imaginary parts jointly determine the phase velocity $\omega/\,\mathrm{Re}(k)$ and attenuation $-\,\mathrm{Im}(k)$ with $k = \omega/c$.

6. The Generalized Maxwell body supplies a causal, thermodynamically admissible $\tilde{\mu}(\omega)$ whose $Q(\omega)$, (11), can be made nearly constant over a chosen band by fitting the weights a_l.

The single-layer check reproduces the classical Love dispersion relation exactly, and $\det \mathbf{G} = 1$ provides a running numerical diagnostic. See [2] for the broader viscoelastic wave-propagation context and [3] for a closely related treatment of guided waves in viscoelastic media.

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