Dispersion and Attenuation of Love Waves in a Stack of N Isotropic Viscoelastic Layers over a Half-Space: A Thomson-Haskell Propagator Matrix Approach.

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1 Isotropic Elastic case

The 3D equations of motion for an isotropic linear-elastic medium can be written as:

$$
\begin{aligned}
\rho\frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} + f_x \\
\rho\frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} + f_y \\
\rho\frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + f_z
\end{aligned} \tag{1}
$$

The tensor of elastic moduli for an **isotropic medium** is given as:

$$
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{bmatrix}
$$

$$
\begin{aligned}
\sigma_{xx} &= \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{xx} \\
\sigma_{yy} &= \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{yy} \\
\sigma_{zz} &= \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{zz} \\
\sigma_{xy} &= 2\mu\epsilon_{xy} \\
\sigma_{yz} &= 2\mu\epsilon_{yz} \\
\sigma_{zx} &= 2\mu\epsilon_{zx}
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\epsilon_{xx} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x}\right) = \frac{\partial u_x}{\partial x} & \qquad \epsilon_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\
\epsilon_{yy} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y}\right) = \frac{\partial u_y}{\partial y} & \qquad \epsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\
\epsilon_{zz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z}\right) = \frac{\partial u_z}{\partial z} & \qquad \epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)
\end{aligned}
$$

Equation 2 cqn be written generally as:

$$
\boxed{\sigma_{ij} = \lambda\vartheta\delta_{ij} + 2\mu\epsilon_{ij}}
$$

where:

- λ and μ are the Lamé parameters (elastic constants)
- $\vartheta = \epsilon_{kk} = \nabla \cdot \mathbf{u}$ is the dilatation (volumetric strain)
- δ_{ij} is the Kronecker delta
- $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the strain tensor
- u_i the displacement [m]
- σ_{ij} the stress tensor [Pa]
- f_i the source term [N/m^3]
- ϵ_{ij} the strain tensor []
- ρ the density [kg/m^3]

2 Isotropic ViscoElastic

Now, to describe a viscoelastic medium, we need to modify the stress-strain relation because the conservation of momentum is independent of the material behavior. In linear viscoelasticity **the stress depends on the history of the strain rate**. The viscoelastic stress-strain relation can be described by generalizing the purely elastic case by **introducing frequency-dependent complex moduli (or quality factor, Q)** or **time-domain convolution integrals described by the Boltzmann superposition and causality principle**:

$$\sigma(t) = \int_{-\infty}^{t} \Psi(t-\tau)\dot{\epsilon}(\tau)\, d\tau$$

$\Psi(t)$ is the relaxation function.

$$\boxed{\sigma_{ij}(t) = \int_{-\infty}^{t} \Psi_{ijkl}(t-\tau)\dot{\epsilon}_{kl}(\tau)\, d\tau}$$

where $G_{ijkl}(t)$ is the **relaxation tensor** and ε_{kl} is the infinitesimal strain,

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right).$$

Using the time convolution notation:

$$a(t) * b(t) = \int_{-\infty}^{\infty} a(t-\tau)b(\tau)d\tau = \int_{0}^{t} a(t-\tau)b(\tau)d\tau$$

The above can be written compactly as:

$$\boxed{\sigma_{ij} = \psi_{ijkl} * \dot{\varepsilon}_{kl}} \tag{3}$$

For an isotropic viscoelastic medium, the constitutive relation takes the form:

$$\boxed{\sigma_{ij} = \delta_{ij}\left(\psi_\lambda * \dot{\vartheta}\right) + 2\,\psi_\mu * \dot{\varepsilon}_{ij}} \tag{4}$$

3 Viscoelastic Stress Components

From Equation 4, we will calculate stress components:

Normal Stresses ($\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$)

$$\sigma_{xx}(t) = \psi_\lambda(t) * \dot{\vartheta}(t) + 2\psi_\mu(t) * \left(\frac{\partial \dot{u}(t)}{\partial x}\right)$$
$$\sigma_{yy}(t) = \psi_\lambda(t) * \dot{\vartheta}(t) + 2\psi_\mu(t) * \left(\frac{\partial \dot{v}(t)}{\partial y}\right)$$
$$\sigma_{zz}(t) = \psi_\lambda(t) * \dot{\vartheta}(t) + 2\psi_\mu(t) * \left(\frac{\partial \dot{w}(t)}{\partial z}\right)$$

Shear Stresses ($\sigma_{xy}, \sigma_{yz}, \sigma_{xz}$)

$$\sigma_{xy}(t) = 2\psi_\mu(t) * \left[\frac{1}{2}\left(\frac{\partial \dot{u}(t)}{\partial y} + \frac{\partial \dot{v}(t)}{\partial x}\right)\right] = \psi_\mu(t) * \left(\frac{\partial \dot{u}(t)}{\partial y} + \frac{\partial \dot{v}(t)}{\partial x}\right)$$

$$\sigma_{yz}(t) = \psi_\mu(t) * \left(\frac{\partial \dot{v}(t)}{\partial z} + \frac{\partial \dot{w}(t)}{\partial y}\right)$$
$$\sigma_{xz}(t) = \psi_\mu(t) * \left(\frac{\partial \dot{u}(t)}{\partial z} + \frac{\partial \dot{w}(t)}{\partial x}\right)$$

Summary and Final Notes:

Assembling all the components, we arrive at the complete equation:

$$\sigma_{xx} = \psi_\lambda * \dot{\vartheta} + 2\psi_\mu * \frac{\partial \dot{u}}{\partial x} \qquad \sigma_{xy} = \psi_\mu * \left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right)$$
$$\sigma_{yy} = \psi_\lambda * \dot{\vartheta} + 2\psi_\mu * \frac{\partial \dot{v}}{\partial y} \qquad \sigma_{yz} = \psi_\mu * \left(\frac{\partial \dot{v}}{\partial z} + \frac{\partial \dot{w}}{\partial y}\right)$$
$$\sigma_{zz} = \psi_\lambda * \dot{\vartheta} + 2\psi_\mu * \frac{\partial \dot{w}}{\partial z} \qquad \sigma_{xz} = \psi_\mu * \left(\frac{\partial \dot{u}}{\partial z} + \frac{\partial \dot{w}}{\partial x}\right)$$

Physical Meaning of the Relaxation Functions:

- $\psi_\mu(t)$: The **shear relaxation modulus**. It describes the time-dependent stress response to a step change in shear strain. It controls the dissipation of S-waves.
- $\psi_\lambda(t)$: This function, along with $\psi_\mu(t)$, governs the relaxation of volumetric stress. It controls the dissipation of P-waves.

In the frequency domain, these convolutions become simple multiplications, and the complex moduli derived from $\psi_\lambda(\omega)$ and $\psi_\mu(\omega)$ define the frequency-dependent velocities and attenuation (quality factors Q_P and Q_S) of the medium.

The constitutive equations in the frequency domain become

$$\tilde{\sigma}_{xx} = i\omega\tilde{\psi}_\lambda\tilde{\vartheta} + 2i\omega\tilde{\psi}_\mu\frac{\partial\tilde{u}}{\partial x}, \qquad \tilde{\sigma}_{xy} = i\omega\tilde{\psi}_\mu\left(\frac{\partial\tilde{u}}{\partial y} + \frac{\partial\tilde{v}}{\partial x}\right),$$
$$\tilde{\sigma}_{yy} = i\omega\tilde{\psi}_\lambda\tilde{\vartheta} + 2i\omega\tilde{\psi}_\mu\frac{\partial\tilde{v}}{\partial y}, \qquad \tilde{\sigma}_{yz} = i\omega\tilde{\psi}_\mu\left(\frac{\partial\tilde{v}}{\partial z} + \frac{\partial\tilde{w}}{\partial y}\right),$$
$$\tilde{\sigma}_{zz} = i\omega\tilde{\psi}_\lambda\tilde{\vartheta} + 2i\omega\tilde{\psi}_\mu\frac{\partial\tilde{w}}{\partial z}, \qquad \tilde{\sigma}_{xz} = i\omega\tilde{\psi}_\mu\left(\frac{\partial\tilde{u}}{\partial z} + \frac{\partial\tilde{w}}{\partial x}\right).$$

$$\tilde{\sigma}_{xx} = \lambda(\omega)\tilde{\vartheta} + 2\mu(\omega)\frac{\partial\tilde{u}}{\partial x}, \qquad \tilde{\sigma}_{xy} = \mu(\omega)\left(\frac{\partial\tilde{u}}{\partial y} + \frac{\partial\tilde{v}}{\partial x}\right),$$
$$\tilde{\sigma}_{yy} = \lambda(\omega)\tilde{\vartheta} + 2\mu(\omega)\frac{\partial\tilde{v}}{\partial y}, \qquad \tilde{\sigma}_{yz} = \mu(\omega)\left(\frac{\partial\tilde{v}}{\partial z} + \frac{\partial\tilde{w}}{\partial y}\right),$$
$$\tilde{\sigma}_{zz} = \lambda(\omega)\tilde{\vartheta} + 2\mu(\omega)\frac{\partial\tilde{w}}{\partial z}, \qquad \tilde{\sigma}_{xz} = \mu(\omega)\left(\frac{\partial\tilde{u}}{\partial z} + \frac{\partial\tilde{w}}{\partial x}\right).$$

where the complex moduli are given by:

$$\lambda(\omega) = i\omega\psi_\lambda(\omega) = \int_{-\infty}^{\infty} \dot{\psi}_\lambda(\omega)e^{-i\omega t}\,dt \tag{5}$$

$$\mu(\omega) = i\omega\psi_\mu(\omega) = \int_{-\infty}^{\infty} \dot{\psi}_\mu(\omega)e^{-i\omega t}\,dt \tag{6}$$

2D SH wave in ViscoElastic Media

SH Wave Configuration

For SH (Shear Horizontal) waves:

1. Particle motion in y-direction
2. Propagation in x-direction
3. Variation in z-direction
4. Only non-zero displacement: $u_y(x, z, t)$
5. Only non-zero stresses: σ_{xy}, σ_{zy}

So, constitutive relations are:

$$\sigma_{xy} = \psi_\mu * \left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right) = \psi_\mu * \frac{\partial \dot{v}}{\partial x}$$

$$\sigma_{yz} = \psi_\mu * \left(\frac{\partial \dot{v}}{\partial z} + \frac{\partial \dot{w}}{\partial y}\right) = \psi_\mu * \frac{\partial \dot{v}}{\partial z}$$

In Frequency Domain (Fourier Transform)

$$\tilde{\sigma}_{xy} = \mu(\omega)\left(\frac{\partial \tilde{v}}{\partial x}\right)$$

$$\tilde{\sigma}_{yz} = \mu(\omega)\left(\frac{\partial \tilde{v}}{\partial z}\right)$$

Complex Shear Modulus

The complex shear modulus can be expressed as:

$$\tilde{\mu}(\omega) = \mu_1(\omega) + i\mu_2(\omega) = \mu_R(\omega) + i\mu_I(\omega)$$

Alternatively:

$$\tilde{\mu}(\omega) = \left[\frac{i\mu\omega\eta}{i\omega\eta + \mu}\right]$$

This is the complex shear modulus using the Maxwell Model

where:

- μ: Spring constant in mechanical models
- η: Dashpot viscosity

4 Wave Equation in Viscoelastic Media

Equation of Motion

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial z} = \rho \frac{\partial^2 u_y}{\partial t^2}$$

Substituting viscoelastic constitutive relations:

$$\frac{\partial}{\partial x}\left(\tilde{\mu}(\omega)\frac{\partial \tilde{u}_y}{\partial x}\right)+\frac{\partial}{\partial z}\left(\tilde{\mu}(\omega)\frac{\partial \tilde{u}_y}{\partial z}\right)=\rho\frac{\partial^2 u_y}{\partial t^2}$$

For homogeneous viscoelastic medium:

$$\tilde{\mu}(\omega)\left(\frac{\partial^2 \tilde{u}_y}{\partial x^2}+\frac{\partial^2 \tilde{u}_y}{\partial z^2}\right)=\rho\frac{\partial^2 u_y}{\partial t^2}$$

$$\boxed{\tilde{\mu}(\omega)\left(\frac{\partial^2 \tilde{u}_y}{\partial x^2}+\frac{\partial^2 \tilde{u}_y}{\partial z^2}\right)=-\rho\omega^2\tilde{u}_y}$$

5 Plane Wave Solution

Assume plane wave solution:

$$u_y(x,z,\omega)=A(z)e^{ikx}$$

Compute derivatives:

$$\frac{\partial^2 \tilde{u}_y}{\partial x^2}=-k^2A(z)e^{ikx},\qquad \frac{\partial^2 \tilde{u}_y}{\partial z^2}=A''(z)e^{ikx}.$$

Substituting into the given equation

$$\tilde{\mu}(\omega)\left(\frac{\partial^2 \tilde{u}_y}{\partial x^2}+\frac{\partial^2 \tilde{u}_y}{\partial z^2}\right)=-\rho\omega^2,$$

we obtain

$$\tilde{\mu}(\omega)\big(A''(z)-k^2A(z)\big)e^{ikx}=-\rho\omega^2A(z)e^{ikx}$$

Dividing both sides by e^{ikx} gives

$$\tilde{\mu}(\omega)\big(A''(z)-k^2A(z)\big)=-\rho\omega^2A(z)$$

$$\tilde{\mu}(\omega)\big(A''-k^2A\big)=-\rho\omega^2A.$$

Simplifying,

$$A''+\left(\frac{\rho\omega^2}{\tilde{\mu}(\omega)}-k^2\right)A=0.$$

Let

$$q^2 := \frac{\omega^2}{\beta^2(\omega)} - k^2, \qquad q = \sqrt{\frac{\omega^2}{\beta^2(\omega)} - k^2}, \qquad \beta(\omega) = \sqrt{\frac{\tilde{\mu}(\omega)}{\rho}}$$

Then the general solution is

$$A(z) = \begin{cases} C_1 \cos(qz) + C_2 \sin(qz), & \text{if } q^2 > 0, \\ C_1 e^{iqz} + C_2 e^{-iqz}, & \text{if } q^2 < 0. \end{cases}$$

$$u_y(x, z, \omega) = (Ae^{ik\,r_\beta z} + Be^{-ik\,r_\beta z})e^{ikx}$$

$$r_\beta = \sqrt{\frac{c^2}{\beta^2(\omega)} - 1}$$

6 General Solution for Love Wave in ViscoElastic Medium

For Love waves in a homogeneous layer, the general solution is:

$$u_y(x, z, \omega) = (Ae^{ik\,r_\beta z} + Be^{-ik\,r_\beta z})e^{ikx}$$

$$r_\beta = \sqrt{\frac{c^2}{\beta^2(\omega)} - 1}$$

$$\tilde{\sigma}_{yz} = \mu(\omega) \left(\frac{\partial \tilde{u}_y}{\partial z} \right)$$

$$\boxed{\frac{\partial u_y}{\partial z} = i\mu k r_\beta \left(Ae^{ikr_\beta z} - Be^{-ikr_\beta z} \right) e^{ikx}.}$$

We can write this in matrix form as a **state vector**:

$$\begin{bmatrix} \tilde{u}_y(z) \\ \tilde{\sigma}_{yz}(z) \end{bmatrix} = \begin{bmatrix} e^{ikr_\beta z} & e^{-ikr_\beta z} \\ i\mu k r_\beta e^{ikr_\beta z} & -i\mu k r_\beta e^{-ikr_\beta z} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \tag{7}$$

7 Thomson-Haskell Propagator Matrix Method for Love Wave Dispersion Analysis

Thomson-Haskell propagator matrix method: is a frequency-domain method for plane waves propagating in a multilayered half-space. A layer-by-layer solution, used for body-wave propagation and surface-wave dispersion problems [2]. The is the standard and most elegant approach for multi-layered media. Now, let's reformulate this problem using propagator matrices.

The Core Problem and Idea for Love Waves

Problem: Calculate the dispersion and attenuation of Love waves propagating in a stack of N horizontal, viscoelastic layers over a semi-infinite viscoelastic half-space.

Core Idea (Propagator Matrix): The state of the SH wavefield at any depth z is described by a **State Vector** (7) containing the relevant continuous field quantities (displacement and shear stress). The **Layer Propagator Matrix** relates the state vector at the top of a layer to its value at the bottom. By successively propagating the state vector from the half-space to the free surface and applying the appropriate boundary conditions, the dispersion equation for the layered medium can be derived.

The Foundation: State Vector and Field Matrix for SH Waves

For SH waves, the motion is purely in the y-direction (transverse to the propagation direction x and depth z).

The **State Vector**, $\mathbf{f}(z)$, for SH waves is:

$$\mathbf{f}(z) = \begin{bmatrix} u_y(z) \\ \sigma_{yz}(z) \end{bmatrix}$$

Where:

- $u_y(z)$: Amplitude of horizontal displacement.
- $\sigma_{yz}(z)$: Shear stress component.

The general solution within a homogeneous, isotropic layer j is a superposition of upgoing and downgoing plane waves:

$$u_y^{(j)}(z) = A_j e^{i\nu_j(z-z_{j-1})} + B_j e^{-i\nu_j(z-z_{j-1})}$$

$$\sigma_{yz}^{(j)}(z) = \mu_j \frac{\partial u_y^{(j)}}{\partial z} = i\mu_j\nu_j \left(A_j e^{i\nu_j(z-z_{j-1})} - B_j e^{-i\nu_j(z-z_{j-1})}\right)$$

where:

- $\nu_j = k\sqrt{(c/\beta_j)^2 - 1}$ is the vertical wavenumber in layer j (can be real or complex).
- A_j is the amplitude of the **upgoing** wave.
- B_j is the amplitude of the **downgoing** wave.
- μ_j is the complex shear modulus of layer j (incorporating viscoelasticity).
- $k = \omega/c$ is the horizontal wavenumber.

We define the **Amplitude Vector**, $\mathbf{a}_j$:

$$\mathbf{a}_j = \begin{bmatrix} A_j \\ B_j \end{bmatrix}$$

The mathematical link between the state vector and the amplitude vector is given by the **Field Matrix**, $\mathbf{E}_j(z)$:

$$\mathbf{f}(z) = \mathbf{E}_j(z)\mathbf{a}_j$$

Explicitly, this is:

$$\begin{bmatrix} u_y(z) \\ \sigma_{yz}(z) \end{bmatrix} = \begin{bmatrix} e^{i\nu_j(z-z_{j-1})} & e^{-i\nu_j(z-z_{j-1})} \\ i\mu_j\nu_j e^{i\nu_j(z-z_{j-1})} & -i\mu_j\nu_j e^{-i\nu_j(z-z_{j-1})} \end{bmatrix} \begin{bmatrix} A_j \\ B_j \end{bmatrix}$$

Derivation of the Layer Propagator Matrix for SH Waves

We want to relate the state vector at the top of a layer ($z = z_t = z_{j-1}$) to the state vector at the bottom ($z = z_b = z_j$).

1. **State at the Bottom:** $\mathbf{f}_{\text{bottom}} = \mathbf{E}_j(z_b)\mathbf{a}_j$. We can solve for the amplitude vector:

   $$\mathbf{a}_j = \mathbf{E}_j^{-1}(z_b)\mathbf{f}_{\text{bottom}}$$

   The inverse of the 2x2 field matrix is straightforward to compute:

$$\mathbf{E}_j^{-1}(z) = \frac{1}{2}\begin{bmatrix} e^{-i\nu_j(z-z_{j-1})} & -\frac{i}{\mu_j\nu_j}e^{-i\nu_j(z-z_{j-1})} \\ e^{i\nu_j(z-z_{j-1})} & \frac{i}{\mu_j\nu_j}e^{i\nu_j(z-z_{j-1})} \end{bmatrix}$$

2. **State at the Top:** $\mathbf{f}_{\text{top}} = \mathbf{E}_j(z_t)\mathbf{a}_j$.

3. **Connect Top to Bottom:** Substitute the expression for $\mathbf{a}_j$:

$$\mathbf{f}_{\text{top}} = \mathbf{E}_j(z_t)\left[\mathbf{E}_j^{-1}(z_b)\mathbf{f}_{\text{bottom}}\right] = \underbrace{\mathbf{E}_j(z_t)\mathbf{E}_j^{-1}(z_b)}_{\mathbf{T}_j}\mathbf{f}_{\text{bottom}}$$

We define the **Layer Propagator Matrix, $\mathbf{T}_j$**:

$$\mathbf{T}_j = \mathbf{E}_j(z_t)\mathbf{E}_j^{-1}(z_b)$$

Let's compute this explicitly. Set the local coordinate so the top of the layer is at $z' = 0$ and the bottom is at $z' = h_j$. Thus $z_t = 0$, $z_b = h_j$.

$$\mathbf{E}_j(0) = \begin{bmatrix} 1 & 1 \\ i\mu_j\nu_j & -i\mu_j\nu_j \end{bmatrix}$$

$$\mathbf{E}_j(h_j) = \begin{bmatrix} e^{i\nu_j h_j} & e^{-i\nu_j h_j} \\ i\mu_j\nu_j e^{i\nu_j h_j} & -i\mu_j\nu_j e^{-i\nu_j h_j} \end{bmatrix}$$

The product $\mathbf{T}_j = \mathbf{E}_j(0)\mathbf{E}_j^{-1}(h_j)$ simplifies to (using hyperbolic functions $\cosh(x) = (e^x + e^{-x})/2, \quad \sinh(x) = (e^x - e^{-x})/2$):

$$\boxed{\mathbf{T}_j = \begin{bmatrix} \cos(\nu_j h_j) & \frac{\sin(\nu_j h_j)}{\mu_j\nu_j} \\ -\mu_j\nu_j\sin(\nu_j h_j) & \cos(\nu_j h_j) \end{bmatrix}}$$

This is the **Thomson-Haskell propagator matrix** for Love waves!

Physical Meaning of $\mathbf{T}_j$: This matrix is a property of the layer. If we know the displacements and stresses at the bottom, we can find them at the top by simply multiplying by $\mathbf{T}_j$. It "propagates" the SH wave solution upwards through the layer.

Building for General N-Layer System

If we have a $\mathbf{N}$ number of layers over a half-space (Layer $\mathbf{n}+1$). The interfaces are at depths $z_1, z_2, z_3, z_4, \ldots z_n$. The thickness of layer j is h_j.

At each interface, the state vector is continuous (welded contact):

$$\mathbf{f}^{(j)}_{\text{bottom}} = \mathbf{f}^{(j+1)}_{\text{top}}$$

Using the propagator matrix for each layer:

$$\mathbf{f}^{(j)}_{\text{top}} = \mathbf{T}_j \mathbf{f}^{(j)}_{\text{bottom}} = \mathbf{T}_j \mathbf{f}^{(j+1)}_{\text{top}}$$

We can chain these relations from the half-space up to the top layer:

$$\mathbf{f}^{(1)}_{\text{top}} = \mathbf{T}_1 \mathbf{f}^{(2)}_{\text{top}} = \mathbf{T}_1 \mathbf{T}_2 \mathbf{f}^{(3)}_{\text{top}} = \mathbf{T}_1 \mathbf{T}_2 \mathbf{T}_3 \mathbf{f}^{(4)}_{\text{top}} = \mathbf{T}_1 \mathbf{T}_2 \mathbf{T}_3 \dots \mathbf{T}_n \mathbf{f}^{(n+1)}_{\text{top}}$$

Let's define the **Global Propagator Matrix**, $\mathbf{G}$:

$$\mathbf{G} = \mathbf{T}_1 \mathbf{T}_2 \mathbf{T}_3 \mathbf{T}_4 \mathbf{T}_5 \dots \mathbf{T}_n$$

So the final relationship is:

$$\boxed{\mathbf{f}_{\text{surface}} = \mathbf{G}\, \mathbf{f}_{\text{halfspace-top}}}$$

Where:

- $\mathbf{f}_{\text{surface}} = \mathbf{f}^{(1)}_{\text{top}}$ is the state vector at the free surface ($z = 0$).
- $\mathbf{f}_{\text{halfspace-top}} = \mathbf{f}^{(n+1)}_{\text{top}}$ is the state vector at the top of the half-space.

Applying Boundary Conditions and Finding Love Waves

Boundary Condition 1: Free Surface (z=0)

At the free surface, the shear stress is zero.

$$\mathbf{f}_{\text{surface}} = \begin{bmatrix} u_y(0) \\ 0 \end{bmatrix}$$

Boundary Condition 2: Radiation Condition in the Half-Space ($z \geq z_n$)

In the half-space (Layer n+1), the solution must be purely **downgoing** and **evanescent**. There can be no wave returning from infinity, so the amplitude of the upgoing wave $A_{n+1} = 0$.

The general solution in the half-space is:

$$u_y^{(n+1)}(z) = \left(A_{n+1}e^{i\nu_{n+1}z} + B_{n+1}e^{-i\nu_{n+1}z}\right) e^{ikx}$$

$$\sigma_{yz}^{(n+1)}(z) = i\mu_{n+1}\nu_{n+1}\left(A_{n+1}e^{i\nu_{n+1}(z-z_n)} - B_{n+1}e^{-i\nu_{n+1}(z-z_n)}\right) e^{ikx}$$

Since $A_{n+1} = 0$:

$$u_y^{(n+1)}(z) = B_{n+1}\, e^{-i\nu_{n+1}(z-z_n)}$$

$$\sigma_{yz}^{(n+1)}(z) = -i\mu_{n+1}\nu_{n+1}B_{n+1}e^{-i\nu_{n+1}(z-z_n)}$$

Therefore, at the top of the half-space ($z = z_n$), the state vector is:

$$\mathbf{f}_{\text{halfspace-top}} = \begin{bmatrix} 1 \\ -i\mu_{n+1}\nu_{n+1} \end{bmatrix} B_{n+1}$$

We can write this as:

$$\mathbf{f}_{\text{halfspace-top}} = \mathbf{V}\, B_{n+1}, \quad \text{where} \quad \mathbf{V} = \begin{bmatrix} 1 \\ -i\mu_{n+1}\nu_{n+1} \end{bmatrix}$$

Here, $\mathbf{V}$ is the **boundary matrix** for the half-space.

Formulating the Dispersion Equation

Substitute the half-space condition into the **global propagation** relation:

$$\mathbf{f}_{\text{surface}} = \mathbf{G}\,\mathbf{f}_{\text{halfspace-top}} = \mathbf{G}\,\mathbf{V}\,B_{n+1}$$

Write this out:

$$\begin{bmatrix} u_y(0) \\ 0 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -i\mu_{n+1}\nu_{n+1} \end{bmatrix} B_{n+1} = \begin{bmatrix} G_{11} - i\mu_{n+1}\nu_{n+1}G_{12} \\ G_{21} - i\mu_{n+1}\nu_{n+1}G_{22} \end{bmatrix} B_{n+1}$$

This gives us two equations:

1. $u_y(0) = (G_{11} - i\mu_{n+1}\nu_{n+1}G_{12})B_{n+1}$
2. $0 = (G_{21} - i\mu_{n+1}\nu_{n+1}G_{22})B_{n+1}$

For a non-trivial solution ($B_{n+1} \neq 0$), the second equation must be zero. This is our **Dispersion Equation**:

$$\boxed{D(\omega, c) = G_{21}(\omega, c) - i\mu_{n+1}(\omega)\nu_{n+1}(\omega, c)\, G_{22}(\omega, c) = 0}$$

Summary and Numerical Solution for 5 Layers Over a Half-space

To find the Love wave modes for the 5-layer system:

The dispersion equation is given as:

$$D(\omega, c) = G_{21}(\omega, c) - i\mu_6(\omega)\nu_6(\omega, c)\, G_{22}(\omega, c) = 0$$

1. **For a given frequency** ω and a trial complex phase velocity c, calculate the vertical wavenumber ν_j for each of the 5 layers and the half-space.

$$\nu_j = \frac{\omega}{c}\sqrt{\left(\frac{c}{\beta_j}\right)^2 - 1}$$

   (Ensure the branch is chosen so $\text{Im}(\nu_6) > 0$ for decay in the half-space).

2. **For each layer** j, calculate its propagator matrix $\mathbf{T}_j$:

$$\mathbf{T}_j = \begin{bmatrix} \cos(\nu_j h_j) & \frac{\sin(\nu_j h_j)}{\mu_j \nu_j} \\ -\mu_j \nu_j \sin(\nu_j h_j) & \cos(\nu_j h_j) \end{bmatrix}$$

3. **Multiply the matrices** to get the global propagator:

$$\mathbf{G} = \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3\mathbf{T}_4\mathbf{T}_5$$

4. **Evaluate the dispersion function:**

$$D(\omega, c) = G_{21} - i\mu_6\nu_6 G_{22}$$

5. **Search for the roots** $D(\omega, c) = 0$. Each root $c(\omega)$ for which $\beta_1 < \text{Re}(c) < \beta_6$ is a valid Love wave mode. The real part of c gives the phase velocity, and the imaginary part describes the attenuation due to viscoelasticity.

6. Repeat over frequency range to get dispersion curves $c(\omega)$

This process systematically applies the **Thomson-Haskell method** to the 5-layer Love wave problem, reducing the complex boundary value problem to a robust numerical root-finding exercise.

Calculating all elements of the global propagator matrix, G

Calculating all elements of the global propagator matrix $\mathbf{G}$ is the core computational step.

1. Mathematical Definition

For an N-layer system:

$$\mathbf{G} = \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3\mathbf{T}_4\mathbf{T}_5 \dots \mathbf{T}_N$$

where each layer matrix is:

$$\mathbf{T}_j = \begin{pmatrix} \cos(\nu_j h_j) & \frac{\sin(\nu_j h_j)}{\mu_j \nu_j} \\ -\mu_j \nu_j \sin(\nu_j h_j) & \cos(\nu_j h_j) \end{pmatrix}$$

2. Step-by-Step Multiplication

Let's multiply these matrices step by step. We'll show both the general pattern and specific element calculations.

Step 1: Define Layer Matrices Explicitly

For layer j, let:

$$\mathbf{T}_j = \begin{pmatrix} C_j & S'_j \\ -S_j & C_j \end{pmatrix}$$

where:

$$C_j = \cos(\nu_j h_j), \qquad S_j = \mu_j \nu_j \sin(\nu_j h_j), \qquad S'_j = \frac{\sin(\nu_j h_j)}{\mu_j \nu_j}$$

Important Identity:

$$S_j \cdot S'_j = \sin^2(\nu_j h_j)$$

Step 2: Multiply First Two Layers

Let $\mathbf{G}^{(2)} = \mathbf{T}_1\mathbf{T}_2$.

$$\mathbf{G}^{(2)} = \begin{pmatrix} C_1 & S_1' \\ -S_1 & C_1 \end{pmatrix} \begin{pmatrix} C_2 & S_2' \\ -S_2 & C_2 \end{pmatrix}$$

Element-by-element calculation:

$$\begin{aligned} G_{11}^{(2)} &= C_1 C_2 - S_1' S_2, \\ G_{12}^{(2)} &= C_1 S_2' + S_1' C_2, \\ G_{21}^{(2)} &= -S_1 C_2 - C_1 S_2, \\ G_{22}^{(2)} &= -S_1 S_2' + C_1 C_2. \end{aligned}$$

Thus:

$$\mathbf{G}^{(2)} = \begin{pmatrix} C_1 C_2 - S_1' S_2 & C_1 S_2' + S_1' C_2 \\ -S_1 C_2 - C_1 S_2 & C_1 C_2 - S_1 S_2' \end{pmatrix}$$

Step 3: Multiply with Third Layer

Let $\mathbf{G}^{(3)} = \mathbf{G}^{(2)} \mathbf{T}_3$.

$$\mathbf{G}^{(3)} = \begin{pmatrix} G_{11}^{(2)} & G_{12}^{(2)} \\ G_{21}^{(2)} & G_{22}^{(2)} \end{pmatrix} \begin{pmatrix} C_3 & S_3' \\ -S_3 & C_3 \end{pmatrix}$$

Elements:

$$\begin{aligned} G_{11}^{(3)} &= G_{11}^{(2)} C_3 - G_{12}^{(2)} S_3, \\ G_{12}^{(3)} &= G_{11}^{(2)} S_3' + G_{12}^{(2)} C_3, \\ G_{21}^{(3)} &= G_{21}^{(2)} C_3 - G_{22}^{(2)} S_3, \\ G_{22}^{(3)} &= G_{21}^{(2)} S_3' + G_{22}^{(2)} C_3. \end{aligned}$$

Step 4: General Recursive Formula

We can see a pattern! For $\mathbf{G}^{(k)} = \mathbf{G}^{(k-1)} \mathbf{T}_k$:

$$\begin{aligned} G_{11}^{(k)} &= G_{11}^{(k-1)} C_k - G_{12}^{(k-1)} S_k, \\ G_{12}^{(k)} &= G_{11}^{(k-1)} S_k' + G_{12}^{(k-1)} C_k, \\ G_{21}^{(k)} &= G_{21}^{(k-1)} C_k - G_{22}^{(k-1)} S_k, \\ G_{22}^{(k)} &= G_{21}^{(k-1)} S_k' + G_{22}^{(k-1)} C_k. \end{aligned}$$

Step 5: Complete 5-Layer Calculation

Apply this recursively.

Initialize: $\mathbf{G}^{(1)} = \mathbf{T}_1$

$$\mathbf{G}^{(1)} = \begin{pmatrix} C_1 & S_1' \\ -S_1 & C_1 \end{pmatrix}$$

Layer 2: $\mathbf{G}^{(2)} = \mathbf{G}^{(1)}\mathbf{T}_2$

$$\begin{aligned} G_{11}^{(2)} &= C_1C_2 - S_1'S_2, \\ G_{12}^{(2)} &= C_1S_2' + S_1'C_2, \\ G_{21}^{(2)} &= -S_1C_2 - C_1S_2, \\ G_{22}^{(2)} &= -S_1S_2' + C_1C_2. \end{aligned}$$

Layer 3: $\mathbf{G}^{(3)} = \mathbf{G}^{(2)}\mathbf{T}_3$

$$\begin{aligned} G_{11}^{(3)} &= G_{11}^{(2)}C_3 - G_{12}^{(2)}S_3, \\ G_{12}^{(3)} &= G_{11}^{(2)}S_3' + G_{12}^{(2)}C_3, \\ G_{21}^{(3)} &= G_{21}^{(2)}C_3 - G_{22}^{(2)}S_3, \\ G_{22}^{(3)} &= G_{21}^{(2)}S_3' + G_{22}^{(2)}C_3. \end{aligned}$$

Layer 4: $\mathbf{G}^{(4)} = \mathbf{G}^{(3)}\mathbf{T}_4$

$$\begin{aligned} G_{11}^{(4)} &= G_{11}^{(3)}C_4 - G_{12}^{(3)}S_4, \\ G_{12}^{(4)} &= G_{11}^{(3)}S_4' + G_{12}^{(3)}C_4, \\ G_{21}^{(4)} &= G_{21}^{(3)}C_4 - G_{22}^{(3)}S_4, \\ G_{22}^{(4)} &= G_{21}^{(3)}S_4' + G_{22}^{(3)}C_4. \end{aligned}$$

Layer 5 (Final): $\mathbf{G} = \mathbf{G}^{(5)} = \mathbf{G}^{(4)}\mathbf{T}_5$

$$\boxed{\begin{aligned} G_{11} &= G_{11}^{(4)}C_5 - G_{12}^{(4)}S_5, \\ G_{12} &= G_{11}^{(4)}S_5' + G_{12}^{(4)}C_5, \\ G_{21} &= G_{21}^{(4)}C_5 - G_{22}^{(4)}S_5, \\ G_{22} &= G_{21}^{(4)}S_5' + G_{22}^{(4)}C_5. \end{aligned}}$$

8 Viscoelastic Model

So far the details of the relaxation function are not defined. Therefore, the objective here is to find a **relaxation function with a frequency-independent $Q(\omega)$-value**. For the application in seismic modelling, it is important that the visco-elastic model can describe a frequency-independent $Q(\omega)$. We can construct viscoelastic models composed of two basic elements.

Generalized Maxwell-model

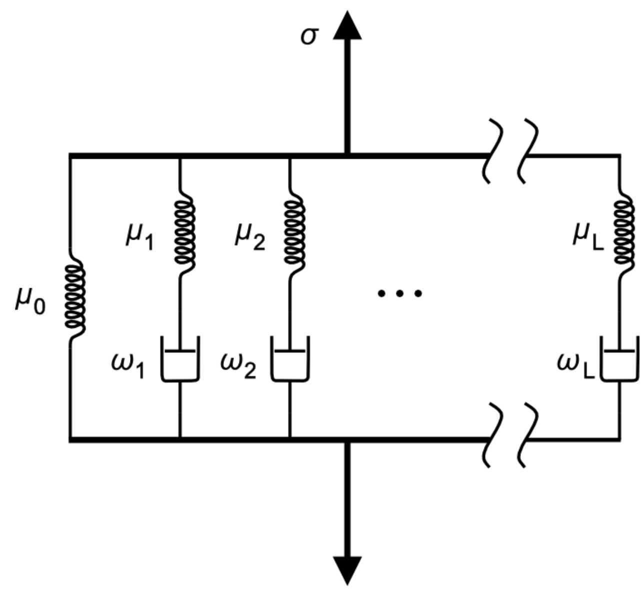

Figure 1: Generalized Maxwell Model

In GMB, we add multiple Maxwell models in parallel, which yields the Generalized Maxwell model or Generalized Maxwell body (GMB), also known as Maxwell-Wiechert model. By the superposition of multiple Maxwell models with different elastic modules μ_l and viscosities η_l, we can achieve a constant Q-value over a given frequency range.

The **Hooke element** (spring), representing the linear elastic medium

$$\sigma_{Hooke} = \mu\epsilon$$

or

$$\epsilon_{Hooke} = \frac{\sigma}{\mu}$$

The **Newton element** (dashpot), representing the viscous damping part with the stress-strainrate relation:

$$\sigma_{Newton} = \eta\dot{\epsilon}$$
$$\text{or}$$
$$\dot{\epsilon}_{Newton} = \frac{\sigma}{\eta}$$

where η denotes the viscosity of the medium.

Because we assemble the Maxwell SLS model with additional L Maxwell bodies in parallel, we have to add the stresses in frequency domain:

$$\tilde{\sigma}_{GMB} = \tilde{\sigma}_{SLSM} + \sum_{l=2}^{L} \tilde{\sigma}_{Maxwell,l}$$

Inserting the stresses

$$\tilde{\sigma}_{SLSM} = \left(\mu_0 + \frac{i\mu_1\omega\eta_1}{i\omega\eta_1 + \mu_1}\right)\tilde{\epsilon}$$
$$\tilde{\sigma}_{Maxwell,l} = \frac{i\mu_l\omega\eta_l}{i\omega\eta_l + \mu_l}\tilde{\epsilon} \tag{8}$$

we have the **frequency-domain stress-strain relation for the GMB:**

$$\tilde{\sigma}_{GMB} = \left(\mu_0 + \frac{i\mu_1\omega\eta_1}{i\omega\eta_1 + \mu_1} + \sum_{l=2}^{L} \frac{i\mu_l\omega\eta_l}{i\omega\eta_l + \mu_l}\right)\tilde{\epsilon}$$

We can move the second term into the sum over the L Maxwell-models:

$$\tilde{\sigma}_{GMB} = \left(\mu_0 + \sum_{l=1}^{L} \frac{i\mu_l\omega\eta_l}{i\omega\eta_l + \mu_l}\right)\tilde{\epsilon}$$

Introducing the **relaxation frequencies:**

$$\omega_l := \frac{\mu_1}{\eta_l}$$

leads to

$$\tilde{\sigma}_{GMB} = \left(\mu_0 + \sum_{l=1}^{L} \frac{i\mu_l\omega}{i\omega + \frac{\mu_1}{\eta_l}}\right)\tilde{\epsilon} = \left(\mu_0 + \sum_{l=1}^{L} \frac{i\mu_l\omega}{i\omega + \omega_l}\right)\tilde{\epsilon}$$

I want to simplify the complex modulus

$$\boxed{\tilde{\mu}_{GMB} = \mu_0 + \sum_{l=1}^{L} \frac{i\mu_l\omega}{i\omega + \omega_l}}$$

First we estimate the relaxed shear modulus:

$$\tilde{\mu}_{GMB,R} = \lim_{\omega\to 0} \tilde{\mu}_{GMB} = \mu_0$$

and unrelaxed shear modulus:

$$\tilde{\mu}_{GMB,U} = \lim_{\omega\to\infty} \tilde{\mu}_{GMB} = \mu_0 + \sum_{l=1}^{L} \frac{i\mu_l\omega}{i\omega\left(1 + \frac{\omega_l}{i\omega}\right)}$$

$$\tilde{\mu}_{GMB,U} = \lim_{\omega\to\infty} \tilde{\mu}_{GMB} = \mu_0 + \sum_{l=1}^{L} \frac{\mu_l}{1 - i\frac{\omega_l}{\omega}}$$

As $\omega \to \infty$, $\frac{\omega_l}{\omega} \to 0$, so:

$$\frac{\mu_l}{1 - i\frac{\omega_l}{\omega}} \to \frac{\mu_l}{1} = \mu_l.$$

$$\tilde{\mu}_{GMB,U} = \lim_{\omega\to\infty} \tilde{\mu}_{GMB} = \mu_0 + \sum_{l=1}^{L} \mu_l$$

With the **modulus defect** or **relaxation of modulus**

$$\delta\mu = \tilde{\mu}_{GMB,U} - \tilde{\mu}_{GMB,R} = \left(\mu_0 + \sum_{l=1}^{L} \mu_l\right) - \mu_0 = \sum_{l=1}^{L} \mu_l$$

For individual mechanisms:

$$\delta\mu_l = \mu_l$$

since each Maxwell body contributes μ_l to the total modulus defect.

Normalization with weights a_l

Write each branch defect as a fraction of the total defect,

$$\delta\mu_l = a_l, \delta\mu$$

where the weights a_l satisfy:

$$\sum_{l=1}^{L} a_l = 1$$

Since $\delta\mu_l = \mu_l$, this gives $\mu_l = a_l, \delta\mu$.

So, each μ_l express as a fraction of the total modulus defect:

$$\mu_l = a_l \delta\mu$$

Verification:

$$\sum_{l=1}^{L} \mu_l = \sum_{l=1}^{L} a_l \delta\mu = \delta\mu \sum_{l=1}^{L} a_l = \delta\mu \cdot 1 = \delta\mu$$

which matches our earlier result.

Substitute $\mu_l = a_l \delta\mu$ into the original expression:

$$\tilde{\mu}_{GMB}(\omega) = \mu_0 + \sum_{l=1}^{L} \frac{i(a_l \delta\mu)\omega}{i\omega + \omega_l}$$

$$\boxed{\tilde{\mu}_{GMB}(\omega) = \mu_0 + \delta\mu \sum_{l=1}^{L} \frac{i a_l \omega}{i\omega + \omega_l}} \tag{9}$$

where μ_0 denotes the **relaxed shear modulus**, $\delta\mu$ the **modulus defect**, L the number of Maxwell bodies, a_l, ω_l **weighting coefficients** and **relaxation frequencies** of the l-th Maxwell body to achieve a constant Q-spectrum, while ω is the circular frequency within the frequency range of the source wavelet.

Final Note:

1. Low-frequency limit: All Maxwell bodies are relaxed $\rightarrow$ only μ_0 remains
2. High-frequency limit: All Maxwell bodies are stiff $\rightarrow$ each contributes μ_l

3. Modulus defect: Difference between high and low frequency limits $= \sum \mu_l$

4. Weight normalization: Distribute total defect among mechanisms with weights a_l

Relaxation Function Derivation: Transformation of Complex Modulus to Time Domain

Deriving the Relaxation Function

We need to transform the complex modulus above to time-domain by inverse Fourier transform leading to the relaxation function.

Given the complex modulus in the frequency domain:

$$\tilde{\mu}_{GMB}(\omega) = \mu_0 + \delta\mu \sum_{l=1}^{L} \frac{i a_l \omega}{i\omega + \omega_l} = \mu_R + \delta\mu \sum_{l=1}^{L} \frac{i a_l \omega}{i\omega + \omega_l}$$

We want to transform this to the **time domain relaxation modulus** $G(t)$.

Relationship between complex modulus and relaxation modulus

In linear viscoelasticity, the complex modulus $\tilde{\mu}(\omega)$ is related to the relaxation modulus $\Psi(t)$ via:

$$\tilde{\mu}(\omega) = i\omega\, \mathcal{F}[\Psi(t)](\omega)$$

where $\mathcal{F}[\Psi(t)](\omega) = \int_0^\infty \Psi(t) e^{-i\omega t} dt$ is the Fourier transform (for causal $\Psi(t)$).

$$\tilde{\mu}(\omega) = i\omega \int_0^\infty \Psi(t) e^{-i\omega t} dt$$

$$\tilde{\mu}(\omega) = i\omega\, \widehat{\Psi}(\omega)$$

This means $\tilde{\mu}(\omega)$ is the Fourier transform of the derivative of $\Psi(t)$, or equivalently:

$$\frac{\tilde{\mu}(\omega)}{i\omega} = \int_0^\infty \Psi(t) e^{-i\omega t} dt$$

Thus, $\Psi(t)$ is the inverse Fourier transform of $\tilde{\mu}(\omega)/(i\omega)$.

So to get $\Psi(t)$, we take the inverse Fourier transform: '

$$\Psi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{\mu}(\omega)}{i\omega} e^{i\omega t} d\omega.$$

Rewrite the Complex Modulus

Divide the given expression by $i\omega$:

$$\frac{\tilde{\mu}(\omega)}{i\omega} = \frac{\mu_0}{i\omega} + \delta\mu \sum_{l=1}^{L} \frac{a_l}{i\omega + \omega_l}$$

Switch to Laplace domain

Using the substitution $s = i\omega$ (one-sided Fourier transform)

Relaxation modulus $\Psi(t)$ has Laplace transform $\overline{\Psi}(s)$ with:

$$\tilde{\mu}(\omega) = s\,\overline{\Psi}(s)\Big|_{s=i\omega}.$$

So:

$$\overline{\Psi}(s) = \frac{\tilde{\mu}(s)}{s} = \frac{\mu_0}{s} + \delta\mu \sum_{l=1}^{L} \frac{a_l}{s + \omega_l}$$

where $\overline{\Psi}(s)$ is the Laplace transform of $\Psi(t)$.

Inverse Laplace transform

Taking the inverse Laplace transform term by term:

We know:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad t \geq 0$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s + \omega_l}\right\} = e^{-\omega_l t}$$

Final time-domain expression

Therefore, the relaxation modulus in the time domain is:

$$\boxed{\Psi(t) = \mu_0 + \delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t}} \tag{10}$$

for $t \geq 0$.

$$\boxed{\Psi(t) = \left\{ \mu_0 + \delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t} \right\} \cdot H(t)} \tag{11}$$

where $H(t)$ is the Heaviside step function.

This is the **stress relaxation function** corresponding to the given **complex modulus** 9.

Using the definition of the modulus defect as the difference between the unrelaxed μ_u and relaxed shear modulus μ_0:

$$\delta\mu = \mu_u - \mu_0$$

we can replace the relaxed by the unrelaxed modulus [1, 3]:

$$\boxed{\Psi(t) = \left\{ \mu_u - \delta\mu \sum_{l=1}^{L} a_l \left(1 - e^{-\omega_l t} \right) \right\} \cdot H(t)} \tag{12}$$

So, we are going to use this function in our dispersion numeriacal calculation above.

Transforming the Relaxation Function 12 to Frequency Domain

Given:

$$\Psi(t) = \left\{ \mu_u - \delta\mu \sum_{l=1}^{L} a_l \left(1 - e^{-\omega_l t} \right) \right\} \cdot H(t)$$

where $H(t)$ is the Heaviside step function.

Rewrite:

$$\Psi(t) = \mu_u H(t) - \delta\mu \sum_{l=1}^{L} a_l \left[1 - e^{-\omega_l t} \right] H(t)$$

$$\Psi(t) = \mu_u H(t) - \delta\mu \sum_{l=1}^{L} a_l H(t) + \delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t} H(t)$$

Note: $\sum_{l=1}^{L} a_l$ is just a constant.

Let $A = \sum_{l=1}^{L} a_l$.

Then:

$$\Psi(t) = \mu_u H(t) - \delta\mu \, A \, H(t) + \delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t} H(t)$$

$$\Psi(t) = \left[\mu_u - \delta\mu A\right] H(t) + \delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t} H(t)$$

Fourier transform

We use the Fourier transform definition:

$$\mathcal{F}\{f(t)\}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

We know:

$$\mathcal{F}\{H(t)\}(\omega) = \pi\delta(\omega) + \frac{1}{i\omega}$$

(in the sense of distributions; the $1/(i\omega)$ is interpreted via the Sokhotski–Plemelj formula, often written as $\frac{1}{i\omega} + \pi\delta(\omega)$).

Also:

$$\mathcal{F}\{e^{-\alpha t} H(t)\} = \frac{1}{\alpha + i\omega}, \quad \alpha > 0.$$

Transform of first term

First term: $(\mu_u - \delta\mu A) H(t)$

$$\mathcal{F}\left\{(\mu_u - \delta\mu A) H(t)\right\} = (\mu_u - \delta\mu A) \left[\pi\delta(\omega) + \frac{1}{i\omega}\right]$$

Transform of second term

Second term: $\delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t} H(t)$

$$\mathcal{F}\left\{\delta\mu\, a_l e^{-\omega_l t} H(t)\right\} = \delta\mu\, a_l \cdot \frac{1}{\omega_l + i\omega}$$

So:

$$\mathcal{F}\left\{\delta\mu \sum_{l=1}^{L} a_l e^{-\omega_l t} H(t)\right\} = \delta\mu \sum_{l=1}^{L} \frac{a_l}{\omega_l + i\omega}$$

Combine

$$\boxed{\widetilde{\Psi}(\omega) = (\mu_u - \delta\mu A)\left[\pi\delta(\omega) + \frac{1}{i\omega}\right] + \delta\mu \sum_{l=1}^{L} \frac{a_l}{\omega_l + i\omega}}$$

Simplify the constant A

Recall:

$$A = \sum_{l=1}^{L} a_l$$

We can combine the $1/(i\omega)$ term with the sum over $a_l/(\omega_l + i\omega)$ if desired, but often in rheology or viscoelasticity, they keep it as a sum of Debye terms plus a singular term at $\omega = 0$.

Let's check: The $\frac{1}{i\omega}$ term coefficient is $\mu_u - \delta\mu A$.
But note: sometimes the model is such that $\Psi(t \to \infty) = \mu_u - \delta\mu A = \mu_R$ (relaxed modulus), and $\Psi(0^+) = \mu_u$ (unrelaxed modulus). Indeed, at $t = 0^+$, $e^{-\omega_l t} = 1$, so

$$\Psi(0^+) = \mu_u - \delta\mu \sum_{l=1}^{L} a_l(1 - 1) = \mu_u.$$

At $t \to \infty$, $e^{-\omega_l t} \to 0$, so

$$\Psi(\infty) = \mu_u - \delta\mu \sum_{l=1}^{L} a_l = \mu_u - \delta\mu A.$$

So indeed, $\mu_R = \mu_u - \delta\mu A$.

Thus:

$$\boxed{\widetilde{\Psi}(\omega) = \mu_R \left[\pi\delta(\omega) + \frac{1}{i\omega}\right] + \delta\mu \sum_{l=1}^{L} \frac{a_l}{\omega_l + i\omega}}$$

This is the relaxation function 12 in frequency domain.

Interpret in complex modulus $G^*(\omega)$

In rheology, the complex modulus $G^*(\omega) = i\omega\widetilde{\Psi}(\omega)$ for a stress relaxation modulus $\Psi(t)$.

Let's compute $G^*(\omega)$:

$$G^*(\omega) = i\omega\widetilde{\Psi}(\omega) = i\omega\,\mathcal{F}[\Psi(t)H(t)]$$

$$= i\omega \left[\mu_R \left(\pi\delta(\omega) + \frac{1}{i\omega}\right) + \delta\mu \sum_{l=1}^{L} \frac{a_l}{\omega_l + i\omega}\right]$$

The term $i\omega \cdot \mu_R\pi\delta(\omega) = i\pi\mu_R\omega\delta(\omega) = 0$ because $\omega\delta(\omega) = 0$.

The term $i\omega \cdot \frac{\mu_R}{i\omega} = \mu_R$.

So:

$$G^*(\omega) = \mu_R + \delta\mu \sum_{l=1}^{L} \frac{i\omega a_l}{\omega_l + i\omega}$$

This is a standard generalized Maxwell model form:

$$G^*(\omega) = \mu_R + \sum_{l=1}^{L} \frac{i\omega\delta\mu a_l}{\omega_l + i\omega}$$

with relaxation strengths $\delta\mu a_l$ for mode l.

Relaxation modulus $\Psi(t)$ in frequency domain

$$\boxed{\widetilde{\Psi}(\omega) = \mu_R \left[\pi\delta(\omega) + \frac{1}{i\omega}\right] + \delta\mu \sum_{l=1}^{L} \frac{a_l}{\omega_l + i\omega}}$$

where $\mu_R = \mu_u - \delta\mu \sum_{l=1}^{L} a_l$.

In summary, $i\omega\widetilde{\Psi}(\omega) = i\omega\,\mathcal{F}[\Psi(t)H(t)]$ also give the complex modulus which is a **Generalized**

Maxwell Model (GMB) in this case.

GMB stress-strain relation

We can then inserting the relaxation function into the viscoelastic stress-strain relation:

$$\sigma = \int_{-\infty}^{t} \dot{\Psi}(t-t')\epsilon(t')dt' = \dot{\Psi} * \epsilon$$

we have to take its first time derivative of $\Psi(t)$:

$$\boxed{\Psi(t) = \left\{\mu_u - \delta\mu \sum_{l=1}^{L} a_l \left(1 - e^{-\omega_l t}\right)\right\} \cdot H(t)} \tag{13}$$

$$\boxed{\dot{\Psi}(t) = -\delta\mu \sum_{l=1}^{L} a_l \omega_l e^{-\omega_l t} \cdot H(t) + \left\{\mu_u - \delta\mu \sum_{l=1}^{L} a_l \left(1 - e^{-\omega_l t}\right)\right\} \cdot \delta(t)}$$

This is the **time derivative of the relaxation function** $\Psi(t)$.

But at $t = 0$, $e^{-\omega_l t} = 1$, (since $\delta(t)$ peaks out $t = 0$) so:

$$\mu_u - \delta\mu \sum_{l=1}^{L} a_l(1-1) = \mu_u.$$

Thus the given $\delta(t)$ coefficient simplifies to μ_u, matching our result.

Final derivative

$$\boxed{\dot{\Psi}(t) = -\delta\mu \sum_{l=1}^{L} a_l \omega_l e^{-\omega_l t} H(t) + \mu_u \, \delta(t)}$$

with $\delta\mu = \mu_u - \mu_0$.

Insert into stress-strain relation

$$\sigma(t) = \int_{-\infty}^{t} \dot{\Psi}(t-t')\epsilon(t') \, dt'$$

$$= \mu_u \epsilon(t) - \delta\mu \sum_{l=1}^{L} a_l \omega_l \int_{-\infty}^{t} e^{-\omega_l (t-t')} H(t-t')\epsilon(t') \, dt'$$

Since $H(t - t') = 1$ for $t' \leq t$, we can write:

$$\boxed{\sigma(t) = \mu_u \epsilon(t) - \delta\mu \sum_{l=1}^{L} a_l \omega_l \int_{-\infty}^{t} e^{-\omega_l (t-t')} \epsilon(t')\, dt'}$$

This is the **viscoelastic stress-strain relation** for the GMB model.

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