The Elastodynamic Equation: Helmholtz Decomposition, Body Waves and Homogeneous Media Solutions

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Elastodynamic Equation

The goal is to derive equation (1) below:

$$\boxed{\rho\ddot{\mathbf{u}} = (\lambda + 2\mu)\,\nabla(\nabla\cdot\mathbf{u})\;-\;\mu\,\nabla\times(\nabla\times\mathbf{u})} \tag{1}$$

Starting from the momentum balance and Hooke's law. We will use index notation (Einstein summation) where helpful and also show the vector identity used.

Assumptions

- Small strains, linear elasticity (infinitesimal strain tensor).
- Homogeneous, isotropic solid with constant Lamé parameters λ, μ.
- No body forces.
- $\mathbf{u}(\mathbf{x}, t)$ is displacement; ρ density.

Step 1 — Balance of linear momentum (Cauchy)

$$\rho\,\ddot{u}_i \;=\; \partial_j \sigma_{ij}, \qquad \text{or in vector form} \quad \rho\,\ddot{\mathbf{u}} = \nabla\cdot\boldsymbol{\sigma}.$$

Step 2 — Constitutive law (isotropic Hooke's law)

Strain (infinitesimal):

$$\varepsilon_{kl} = \tfrac{1}{2}(\partial_k u_l + \partial_l u_k).$$

Stress:

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} = \lambda\,\delta_{ij}\,\varepsilon_{kk} + \; 2\mu\,\varepsilon_{ij}.$$

Since $\varepsilon_{kk} = \partial_k u_k$ and $\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, we get

$$\boxed{\sigma_{ij} = \lambda\,\delta_{ij}\,(\partial_k u_k) \;+\; \mu(\partial_i u_j + \partial_j u_i)\;.} \tag{2}$$

Step 3 — Compute $\partial_j \sigma_{ij}$

Differentiate (1) with respect to x_j:

$$\partial_j \sigma_{ij} = \partial_j\big(\lambda\,\delta_{ij}\partial_k u_k\big) + \partial_j\big(\mu(\partial_i u_j + \partial_j u_i)\big).$$

With constant λ, μ (homogeneous medium) and δ_{ij} constant:

$$\partial_j\big(\lambda\,\delta_{ij}\partial_k u_k\big) = \lambda\,\partial_i(\partial_k u_k).$$

For the second term:

$$\partial_j\big(\mu(\partial_i u_j + \partial_j u_i)\big) = \mu\big(\partial_j\partial_i u_j + \partial_j\partial_j u_i\big).$$

So

$$\boxed{\partial_j \sigma_{ij} = \lambda\, \partial_i(\partial_k u_k) + \mu\, \partial_j \partial_i u_j + \mu\, \partial_j \partial_j u_i\ .} \tag{3}$$

Step 4 — Use commutativity of partial derivatives

Since mixed partials commute for smooth fields,

$$\partial_j \partial_i u_j = \partial_i(\partial_j u_j).$$

Define $\nabla \cdot \mathbf{u} = \partial_j u_j$ and Laplacian $\nabla^2 u_i = \partial_j \partial_j u_i$. Then (3) becomes

$$\partial_j \sigma_{ij} = \lambda\, \partial_i(\nabla \cdot \mathbf{u}) + \mu\, \partial_i(\nabla \cdot \mathbf{u}) + \mu\, \nabla^2 u_i$$

$$\partial_j \sigma_{ij} = (\lambda + \mu)\, \partial_i(\nabla \cdot \mathbf{u}) + \mu\, \nabla^2 u_i.$$

So the momentum equation is

$$\rho\, \ddot{u}_i = (\lambda + \mu)\, \partial_i(\nabla \cdot \mathbf{u}) + \mu\, \nabla^2 u_i. \tag{4a}$$

In vector form:

$$\boxed{\rho\, \ddot{\mathbf{u}} = (\lambda + \mu)\, \nabla(\nabla \cdot \mathbf{u}) + \mu\, \nabla^2 \mathbf{u}} \tag{4b}$$

Step 5 — Replace $\nabla^2\mathbf{u}$ using a vector identity

Use the standard identity

$$\boxed{\nabla^2 \mathbf{u} \;=\; \nabla(\nabla \cdot \mathbf{u}) \;-\; \nabla \times (\nabla \times \mathbf{u})\ .} \tag{5}$$

(you may also prove (5) componentwise.)

Substitute (5) into (4b):

$$\rho\,\ddot{\mathbf{u}} = (\lambda+\mu)\,\nabla(\nabla\cdot\mathbf{u}) + \mu\Big[\nabla(\nabla\cdot\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u})\Big].$$

Collect the $\nabla(\nabla\cdot\mathbf{u})$ terms:

$$\rho\,\ddot{\mathbf{u}} = (\lambda+2\mu)\,\nabla(\nabla\cdot\mathbf{u}) \;-\; \mu\,\nabla\times(\nabla\times\mathbf{u}).$$

This is the desired equation (1).

$$\boxed{\rho\,\ddot{\mathbf{u}} \;=\; (\lambda+2\mu)\,\nabla(\nabla\cdot\mathbf{u}) \;-\; \mu\,\nabla\times(\nabla\times\mathbf{u})\;.} \tag{1}$$

Index-notation derivation (compact)

From

$$\rho\ddot{u}_i = \partial_j\sigma_{ij}$$

$$\sigma_{ij} = \lambda\delta_{ij}\partial_k u_k + \mu(\partial_i u_j + \partial_j u_i)$$

Compute $\partial_j\sigma_{ij} = \lambda\partial_i(\partial_k u_k) + \mu\partial_j\partial_j u_i + \mu\partial_j\partial_i u_j$. Rearranging and using $\partial_j\partial_i u_j = \partial_i(\partial_j u_j)$ gives

$$\boxed{\rho\ddot{u}_i = (\lambda+\mu)\partial_i(\partial_j u_j) + \mu\partial_j\partial_j u_i}$$

which is algebraically equivalent to (1) after grouping into $\nabla(\nabla\cdot\mathbf{u})$ and $\nabla^2\mathbf{u}$ and using the identity for $\nabla^2\mathbf{u}$.

Appendix — component proof of identity (5)

Show $(\nabla \times (\nabla \times \mathbf{u}))_i = \partial_i(\partial_j u_j) - \nabla^2 u_i$.

Start with Levi–Civita:

$$(\nabla \times (\nabla \times \mathbf{u}))_i = \epsilon_{ijk}\partial_j\big(\epsilon_{klm}\partial_l u_m\big) = \epsilon_{ijk}\epsilon_{klm}\,\partial_j\partial_l u_m.$$

Use the epsilon–delta identity

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl},$$

so

$$(\nabla \times (\nabla \times \mathbf{u}))_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\partial_l u_m = \partial_j\partial_i u_j - \partial_j\partial_j u_i.$$

But $\partial_j\partial_i u_j = \partial_i(\partial_j u_j) = \partial_i(\nabla \cdot \mathbf{u})$ and $\partial_j\partial_j u_i = \nabla^2 u_i$. Hence

$$\boxed{(\nabla \times (\nabla \times \mathbf{u}))_i = \partial_i(\nabla \cdot \mathbf{u}) - \nabla^2 u_i}$$

which rearranges to equation (5).

General Note

- The first term $(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u})$ is the *irrotational* (dilatational) part $\to$ P waves.
- The second term $-\mu\nabla \times (\nabla \times \mathbf{u})$ is the *solenoidal* (shear) part $\to$ S waves.

2.0 – Helmholtz Decomposition

The **Helmholtz Decomposition** states that any sufficiently smooth vector field $\mathbf{u}$ can be decomposed into a curl-free component and a divergence-free component:

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s = \nabla\phi + \nabla \times \mathbf{\Psi} \tag{2}$$

where:

- ϕ is a **scalar potential**
- $\mathbf{\Psi}$ is a **vector potential**
- $\mathbf{u}_p = \nabla\phi$ is curl-free ($\nabla \times \mathbf{u}_p = 0$)
- $\mathbf{u}_s = \nabla \times \mathbf{\Psi}$ is divergence-free ($\nabla \cdot \mathbf{u}_s = 0$)

Decoupling the Wave Equation

We now substitute the decomposition (2) into the simplified wave equation 1.

$$\boxed{\rho\ddot{\mathbf{u}} = (\lambda + 2\mu)\,\nabla(\nabla \cdot \mathbf{u}) \;-\; \mu\,\nabla \times (\nabla \times \mathbf{u})} \tag{1}$$

Left-Hand Side (Time Derivatives)

$$\rho\ddot{\mathbf{u}} = \rho\frac{\partial^2}{\partial t^2}(\nabla\phi + \nabla \times \mathbf{\Psi}) = \rho\nabla\ddot{\phi} + \rho\nabla \times \ddot{\mathbf{\Psi}} \tag{3}$$

Right-Hand Side (Spatial Derivatives)

$$(\lambda + 2\mu)\,\nabla(\nabla \cdot \mathbf{u}) \;-\; \mu\,\nabla \times (\nabla \times \mathbf{u})$$

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\nabla\phi + \nabla \times \mathbf{\Psi}) = \nabla \cdot (\nabla\phi) + \nabla \cdot (\nabla \times \mathbf{\Psi}) = \nabla^2\phi + 0 \tag{4}$$

$$\nabla \times \mathbf{u} = \nabla \times (\nabla\phi + \nabla \times \mathbf{\Psi}) = \nabla \times (\nabla\phi) + \nabla \times (\nabla \times \mathbf{\Psi}) = 0 + \nabla \times (\nabla \times \mathbf{\Psi}) \tag{5}$$

Substituting (4) and (5) into the right-hand side of equation 1:

$$\begin{aligned}\text{RHS} &= (\lambda+2\mu)\nabla(\nabla^2\phi) - \mu\nabla\times[\nabla\times(\nabla\times\mathbf{\Psi})] \\ &= (\lambda+2\mu)\nabla(\nabla^2\phi) - \mu\nabla\times[\nabla(\nabla\cdot\mathbf{\Psi}) - \nabla^2\mathbf{\Psi}] \end{aligned} \tag{6}$$

We can choose the **Coulomb gauge** ($\nabla\cdot\mathbf{\Psi}=0$), which simplifies (6) to:

$$\text{RHS} = (\lambda+2\mu)\nabla(\nabla^2\phi) + \mu\nabla\times(\nabla^2\mathbf{\Psi}) \tag{7}$$

The Final Decoupled Form

Equating the left-hand side (3) with the right-hand side (7):

$$\nabla\left[(\lambda+2\mu)\nabla^2\phi - \rho\ddot{\phi}\right] + \nabla\times\left[\mu\nabla^2\mathbf{\Psi} - \rho\ddot{\mathbf{\Psi}}\right] = 0 \tag{8}$$

For this sum of a gradient and a curl to be zero everywhere, the terms inside the brackets must each be zero (or at most equal to a constant, which can be ignored for wave solutions):

$$\nabla\left[(\lambda+2\mu)\nabla^2\phi - \rho\ddot{\phi}\right] = 0 \tag{9}$$

$$\nabla\times\left[\mu\nabla^2\mathbf{\Psi} - \rho\ddot{\mathbf{\Psi}}\right] = 0 \tag{10}$$

This leads to two independent, decoupled wave equations.

Final Wave Equations and Speeds

From (9) and (10), we obtain:

P-Wave Equation (Compressional)

$$\frac{1}{\alpha^2}\ddot{\phi} = \nabla^2\phi, \quad \text{where} \quad \alpha = \sqrt{\frac{\lambda+2\mu}{\rho}} \tag{11}$$

- α is the **P-wave speed**.
- Derived from the scalar potential ϕ.
- Particle motion is **parallel** to the direction of propagation ($\mathbf{u}_p = \nabla\phi$).

S-Wave Equation (Shear)

$$\frac{1}{\beta^2}\ddot{\boldsymbol{\Psi}} = \nabla^2 \boldsymbol{\Psi}, \quad \text{where} \quad \beta = \sqrt{\frac{\mu}{\rho}} \tag{12}$$

- β is the **S-wave speed**.
- Derived from the vector potential $\boldsymbol{\Psi}$.
- Particle motion is **perpendicular** to the direction of propagation ($\mathbf{u}_s = \nabla \times \boldsymbol{\Psi}$).

Summary

The Helmholtz decomposition successfully decouples the elastic wave equation into two independent wave types:

- **P-waves** (faster, $\alpha > \beta$, compressional) governed by the scalar potential ϕ.
- **S-waves** (slower, shear) governed by the vector potential $\boldsymbol{\Psi}$.

This derivation elegantly explains the fundamental separation of body waves observed in seismology.

3.0 – Homogeneous Media Solution (Plane vs Spherical Waves)

Plane Wave Solutions and Polarization

We start from the homogeneous isotropic elastodynamic equation *(already established above)*:

$$\boxed{\rho\, \ddot{\mathbf{u}} \;=\; (\lambda + 2\mu)\, \nabla(\nabla \cdot \mathbf{u}) \;-\; \mu\, \nabla \times (\nabla \times \mathbf{u})\;.} \tag{1}$$

We seek harmonic **plane-wave** solutions to the elastodynamic equation.

Step 1 — Plane-wave ansatz

Assume

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{A}\, e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3 - \omega t)} = \mathbf{A}\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

where $\mathbf{k}$ is the wavevector, ω angular frequency, and $\mathbf{A}$ (constant) amplitude / polarization vector.

For this ansatz the differential operators act as:

$$\nabla \mapsto i\mathbf{k}, \qquad \partial_t \mapsto -i\omega, \qquad \nabla^2 \mapsto -|\mathbf{k}|^2.$$

So compute the pieces needed:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= i(\mathbf{k} \cdot \mathbf{A})\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\ \nabla(\nabla \cdot \mathbf{u}) &= -\mathbf{k}(\mathbf{k} \cdot \mathbf{A})\, e^{i(\cdot)} \\ \nabla \times \mathbf{u} &= i(\mathbf{k} \times \mathbf{A})\, e^{i(\cdot)} \\ \nabla \times (\nabla \times \mathbf{u}) &= -\mathbf{k} \times (\mathbf{k} \times \mathbf{A})\, e^{i(\cdot)} \\ \ddot{\mathbf{u}} &= -\omega^2 \mathbf{A}\, e^{i(\cdot)} \end{aligned}$$

(Note that $e^{i(\cdot)}$ is short for $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$.)

Step 2 - Substitute into equation (1) and cancel the common exponential factor

Substitute all terms into (1). After cancelling the common factor $e^{i(\cdot)}$ we get

$$\rho(-\omega^2)\mathbf{A} \;=\; (\lambda + 2\mu)\big[-\mathbf{k}(\mathbf{k} \cdot \mathbf{A})\big] \;-\; \mu\big[-\mathbf{k} \times (\mathbf{k} \times \mathbf{A})\big].$$

Simplify signs:

$$-\,\rho\omega^2 \mathbf{A} \;=\; -(\lambda + 2\mu)\,\mathbf{k}(\mathbf{k} \cdot \mathbf{A}) \;+\; \mu\,\mathbf{k} \times (\mathbf{k} \times \mathbf{A}).$$

Multiply both sides by -1:

$$\rho\omega^2 \mathbf{A} \;=\; (\lambda + 2\mu)\,\mathbf{k}(\mathbf{k} \cdot \mathbf{A}) \;-\; \mu\,\mathbf{k} \times (\mathbf{k} \times \mathbf{A}). \tag{6}$$

Step 3 - Use the vector triple-product identity

Use

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{A}) = \mathbf{k}(\mathbf{k} \cdot \mathbf{A}) - |\mathbf{k}|^2 \mathbf{A}.$$

Plugging this into (6):

$$\begin{aligned}
\rho\omega^2 \mathbf{A} &= (\lambda + 2\mu)\, \mathbf{k}(\mathbf{k} \cdot \mathbf{A}) - \mu\big[\mathbf{k}(\mathbf{k} \cdot \mathbf{A}) - |\mathbf{k}|^2 \mathbf{A}\big] \\
&= (\lambda + 2\mu - \mu)\, \mathbf{k}(\mathbf{k} \cdot \mathbf{A}) + \mu|\mathbf{k}|^2 \mathbf{A} \\
&= (\lambda + \mu)\, \mathbf{k}(\mathbf{k} \cdot \mathbf{A}) + \mu|\mathbf{k}|^2 \mathbf{A}.
\end{aligned}$$

Rearrange:

$$\boxed{(\rho\omega^2 - \mu|\mathbf{k}|^2)\, \mathbf{A} \ = \ (\lambda + \mu)\, \mathbf{k}(\mathbf{k} \cdot \mathbf{A}).} \tag{7}$$

Equation (7) is an algebraic eigenvalue-type equation for $\mathbf{A}$.

Step 4 - Dot equation (7) with k to find a scalar relation

Take the dot product of (7) with $\mathbf{k}$:

$$(\rho\omega^2 - \mu|\mathbf{k}|^2)\, (\mathbf{k} \cdot \mathbf{A}) \ = \ (\lambda + \mu)\, |\mathbf{k}|^2 (\mathbf{k} \cdot \mathbf{A}).$$

Move all terms to one side:

$$\boxed{\big(\rho\omega^2 - (\lambda + 2\mu)|\mathbf{k}|^2\big)\, (\mathbf{k} \cdot \mathbf{A}) = 0} \tag{8}$$

Thus **either**

- $\mathbf{k} \cdot \mathbf{A} = 0$ (transverse polarization), **or**
- $\mathbf{k} \cdot \mathbf{A} \neq 0$ (longitudinal polarization)
- $\rho\omega^2 = (\lambda + 2\mu)|\mathbf{k}|^2$ (longitudinal branch).

We now have two possibilities.

Case A — Longitudinal (P) wave: $\mathbf{k} \cdot \mathbf{A} \neq 0$

If $\mathbf{k} \cdot \mathbf{A} \neq 0$, equation (8) forces

$$\left(\rho\omega^2 - (\lambda + 2\mu)|\mathbf{k}|^2\right) = \frac{0}{(\mathbf{k} \cdot \mathbf{A})} \tag{9}$$

$$\rho\omega^2 = (\lambda + 2\mu)\,|\mathbf{k}|^2$$

Hence

$$\omega = \pm\sqrt{\frac{\lambda + 2\mu}{\rho}}\;|\mathbf{k}|$$

Define the compressional (P) speed

$$\boxed{\alpha \;=\; \sqrt{\frac{\lambda + 2\mu}{\rho}}\,, \qquad \omega = \pm\alpha|\mathbf{k}|}$$

Now, let's show that $\mathbf{A}$ must be parallel to $\mathbf{k}$.

Using equation (7):

$$(\rho\omega^2 - \mu|\mathbf{k}|^2)\,\mathbf{A} \;=\; (\lambda + \mu)\,\mathbf{k}(\mathbf{k} \cdot \mathbf{A}).$$

Remember:

$$\rho\omega^2 = (\lambda + 2\mu)\,|\mathbf{k}|^2$$

$$\left((\lambda + 2\mu)\,|\mathbf{k}|^2 - \mu|\mathbf{k}|^2\right)\;\mathbf{A} \;=\; (\lambda + \mu)\,\mathbf{k}(\mathbf{k} \cdot \mathbf{A})$$

$$\left(\lambda\,|\mathbf{k}|^2 + 2\mu\,|\mathbf{k}|^2 - \mu\,|\mathbf{k}|^2\right)\;\mathbf{A} \;=\; (\lambda + \mu)\,\mathbf{k}(\mathbf{k} \cdot \mathbf{A})$$

$$(\lambda + \mu)\,|\mathbf{k}|^2\,\mathbf{A} \;=\; (\lambda + \mu)\,\mathbf{k}(\mathbf{k}\cdot\mathbf{A})$$

Thus equation (7) becomes

$$\boxed{(\lambda + \mu)|\mathbf{k}|^2\mathbf{A} = (\lambda + \mu)\,\mathbf{k}(\mathbf{k}\cdot\mathbf{A})}$$

Cancel $(\lambda + \mu)$ (nonzero for ordinary solids) and divide by $|\mathbf{k}|^2$:

$$\mathbf{A} = \frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{A})}{|\mathbf{k}|^2}.$$

This means $\mathbf{A}$ equals its projection onto $\mathbf{k}$ — i.e. $\mathbf{A}$ is **parallel to k**. Thus **P-wave polarization is longitudinal (particle motion parallel to propagation)**.

Case B — Transverse (S) wave: $\mathbf{k}\cdot\mathbf{A} = 0$

If $\mathbf{k}\cdot\mathbf{A} = 0$, equation (7) reduces to

$$(\rho\omega^2 - \mu|\mathbf{k}|^2)\,\mathbf{A} = \mathbf{0}.$$

Non-trivial $\mathbf{A}$ requires

$$\rho\omega^2 = \mu|\mathbf{k}|^2,$$

so

$$\omega = \pm\sqrt{\frac{\mu}{\rho}}\,|\mathbf{k}|.$$

Define the shear (S) speed

$$\boxed{\beta \;=\; \sqrt{\frac{\mu}{\rho}}\,, \qquad \omega = \pm\beta|\mathbf{k}|}$$

Because $\mathbf{k}\cdot\mathbf{A} = 0$, the polarization $\mathbf{A}$ is **perpendicular to k** (transverse). There are two independent orthogonal polarizations (S is twofold degenerate).

Compact matrix/eigenvalue viewpoint (short form)

Rewrite (7) dividing by $|\mathbf{k}|^2$:

$$\big(\rho c^2\big)\mathbf{A} = \mu\mathbf{A} + (\lambda+\mu)\,\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{A}), \quad \text{with } c^2 = \frac{\omega^2}{|\mathbf{k}|^2},\ \hat{\mathbf{n}} = \frac{\mathbf{k}}{|\mathbf{k}|}.$$

This is an eigenproblem for the 3×3 matrix $\mu I + (\lambda+\mu)\hat{n}\hat{n}^T$. Its eigenvalues are:

- $\rho c^2 = \lambda + 2\mu$ for eigenvector $\hat{n}$ (longitudinal),
- $\rho c^2 = \mu$ (double) for any vector orthogonal to $\hat{n}$ (two shear polarizations).

Final Notes

- **P (compressional / longitudinal) waves**: $\omega = \pm\alpha|\mathbf{k}|,\ \alpha = \sqrt{\frac{\lambda+2\mu}{\rho}},\ \mathbf{A} \parallel \mathbf{k}$.
- **S (shear / transverse) waves**: $\omega = \pm\beta|\mathbf{k}|,\ \beta = \sqrt{\frac{\mu}{\rho}},\ \mathbf{A} \perp \mathbf{k}$.

Both are nondispersive (phase speed = group speed) in a homogeneous linear elastic medium. That is, same wave velocity in all direction (Isotropy). S-waves are absent if $\mu = 0$ (fluids).

Spherical Wave Solutions

Physical Context and Wave Equation

In this case we are dealing with waves radiating outward from a point source in a homogeneous medium. The governing equation is the standard scalar wave equation derived for the P-wave potential ϕ:

$$\frac{1}{\alpha^2}\ddot{\phi} = \nabla^2\phi$$

Where:

- α is the wave speed (e.g., P-wave speed)
- ∇^2 is the Laplacian operator

- ϕ is the wave field (e.g., velocity potential)

The key point is that the **form of the Laplacian ∇^2 depends on the coordinate system**, which in turn dictates the nature of the solution.

In spherical coordinates (for spherically symmetric waves):

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)$$

Part 1: First Attempt - Why Simple Plane Wave Fails

Try:

$$\phi(r,t) = A\,e^{i(\omega t - kr)}$$

Calculate first derivative:

$$\frac{\partial\phi}{\partial r} = -ikA\,e^{i(\omega t - kr)} = -ik\phi(r,t)$$

Calculate second part:

$$r^2\frac{\partial\phi}{\partial r} = -ikr^2\phi(r,t)$$

Apply outer derivative:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = \frac{\partial}{\partial r}(-ikr^2\phi(r,t))$$

Using product rule:

$$\begin{aligned}
&= -ik\left[2r\phi(r,t) + r^2\frac{\partial\phi(r,t)}{\partial r}\right] \\
&= -ik\left[2r\phi(r,t) + r^2(-ik\phi(r,t))\right] \\
&= -2ikr\phi(r,t) - k^2r^2\phi(r,t)
\end{aligned}$$

Therefore:

$$\nabla^2\phi = \frac{1}{r^2}\left[-2ikr\phi - k^2r^2\phi\right] = -\frac{2ik}{r}\phi - k^2\phi$$

What we need for the wave equation:

$$\nabla^2\phi = -k^2\phi \quad \text{(where } k = \omega/\alpha\text{)}$$

There's a **Problem!**, because we have an extra term!

$$\nabla^2\phi = -k^2\phi - \frac{2ik}{r}\phi$$

$$\boxed{\nabla^2\phi = -\left(k^2 - \frac{2ik}{r}\right)\phi} \quad \text{WRONG!}$$

The extra term $-2ik/r$ means this is NOT a solution. You should note that Amplitude A in the plane wave assumption above does not vary with radial distance r from the source (spherical symmetry).

Part 2: Physical Insight - Why Amplitude Must Decay

Energy conservation argument:

1. **Energy flows outward through spherical surfaces**
2. **Energy density** $\propto |\phi|^2$
3. **Surface area** of sphere $= 4\pi r^2$
4. **Total energy** through sphere must be constant

Therefore:

$$|\phi|^2 \times 4\pi r^2 = \text{constant}$$

This means:

$$|\phi| \propto \frac{1}{r}$$

The amplitude must decay as 1/r! Practically speaking, It also make sense that A should decay with r.

Part 3: Correct Solution with Amplitude Decay - The Exact 3D Spherical Wave Solution

Assuming:

$$\phi(r,t) = A(r)e^{i(\omega t - kr)}$$

where $A(r)$ is a function of r to be determined.

Find first derivative:

$$\frac{\partial \phi}{\partial r} = \frac{dA}{dr}\, e^{i(\omega t - kr)} - ikA\, e^{i(\omega t - kr)}$$

$$\frac{\partial \phi}{\partial r} = \left[\frac{dA}{dr} - ikA\right] e^{i(\omega t - kr)}$$

Multiply by r^2

$$r^2 \frac{\partial \phi}{\partial r} = r^2 \left[\frac{dA}{dr} - ikA\right] e^{i(\omega t - kr)}$$

$$r^2 \frac{\partial \phi}{\partial r} = r^2 \frac{dA}{dr}\, e^{i(\omega t - kr)} - ikA\, r^2 e^{i(\omega t - kr)}$$

Take derivative again (product rule)

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = e^{i(\omega t - kr)} \times \left[2r\frac{dA}{dr} + r^2 \frac{d^2 A}{dr^2} - 2ikrA - ikr^2 \frac{dA}{dr} + (-ik)\left(r^2 \frac{dA}{dr} - ikr^2 A\right)\right]$$

Simplifying:

$$= e^{i(\omega t - kr)} \left[r^2 \frac{d^2 A}{dr^2} + 2r\frac{dA}{dr} - 2ikr^2 \frac{dA}{dr} - 2ikrA - k^2 r^2 A\right]$$

Divide by r^2 to get $\nabla^2 \phi$

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = e^{i(\omega t - kr)} \left[\frac{d^2 A}{dr^2} + \frac{2}{r}\frac{dA}{dr} - 2ik\frac{dA}{dr} - \frac{2ik}{r}A - k^2 A\right]$$

Wave equation requires that:

$$\nabla^2 \phi = -\frac{\omega^2}{\alpha^2}\phi = -k^2 A e^{i(\omega t - kr)}$$

Equating both sides

$$\frac{d^2A}{dr^2} + \frac{2}{r}\frac{dA}{dr} - 2ik\frac{dA}{dr} - \frac{2ik}{r}A - k^2A = -k^2A$$

The k^2A terms cancel:

$$\frac{d^2A}{dr^2} + \frac{2}{r}\frac{dA}{dr} - 2ik\frac{dA}{dr} - \frac{2ik}{r}A = \cancel{-k^2A} + \cancel{k^2A}$$

$$\frac{d^2A}{dr^2} + \frac{2}{r}\frac{dA}{dr} - 2ik\frac{dA}{dr} = \frac{2ik}{r}A \tag{A}$$

This is a form of Cauchy-Euler equation.

Now, we must solve equation (A) to find $A(r)$

Rearrange the equation

$$\frac{d^2A}{dr^2} + \left(\frac{2}{r} - 2ik\right)\frac{dA}{dr} = \frac{2ik}{r}A$$

Since we have both real and imaginary terms in (A), let's look at the **imaginary part** only.

The imaginary terms are:

$$-2ik\frac{dA}{dr} = \frac{2ik}{r}A$$

Divide both sides by $-2ik$:

$$\frac{dA}{dr} = -\frac{A}{r}$$

Solve the differential equation. This is a separable first-order ODE:

$$\frac{dA}{A} = -\frac{dr}{r}$$

Integrate both sides of the differential equation

$$\int \frac{dA}{A} = \int -\frac{dr}{r}$$

$$\ln|A| = -\ln|r| + C_1$$

$$\ln|A| = \ln\left|\frac{1}{r}\right| + C_1$$

Exponentiate

$$A = e^{\ln(1/r)+C_1} = e^{C_1} \cdot \frac{1}{r}$$

Let $C = e^{C_1}$:

$$\boxed{A(r) = \frac{C}{r}}$$

Let's verify whether this satisfies the real part

With $A = C/r$:

- $\dfrac{dA}{dr} = -\dfrac{C}{r^2}$
- $\dfrac{d^2A}{dr^2} = \dfrac{2C}{r^3}$

Substitute into the original equation (A):

$$\frac{2C}{r^3} + \frac{2}{r}\left(-\frac{C}{r^2}\right) - 2ik\left(-\frac{C}{r^2}\right) = \frac{2ik}{r}\left(\frac{C}{r}\right)$$

$$\frac{2C}{r^3} - \frac{2C}{r^3} + \frac{2ikC}{r^2} = \frac{2ikC}{r^2} \quad \checkmark$$

The equation is satisfied!

So, the final answer is:

$$\boxed{A(r) = \frac{C}{r}}$$

Taking $C = 1$ gives us:

$$A(r) = \frac{1}{r}$$

This is the **geometrical spreading factor** for spherical waves in 3D. The amplitude of the wave decays inversely with distance r because the wave's energy is being distributed over an ever-increasing spherical surface area ($4\pi r^2$). This is a fundamental property of waves in 3D. The waveform remains unchanged (dispersionless) during propagation.

Testing for $\nabla^2\phi$

Let's verify this satisfies the wave equation. Assume $A(r) = C/r$. Then:

$$\phi = \frac{C}{r} e^{i(\omega t - kr)}$$

First derivative:

$$\frac{\partial \phi}{\partial r} = -\frac{C}{r^2} e^{i(\omega t - kr)} - ik\frac{C}{r} e^{i(\omega t - kr)} = -C\left(\frac{1}{r^2} + \frac{ik}{r}\right) e^{i(\omega t - kr)}$$

Now compute the key term for the Laplacian:

$$r^2 \frac{\partial \phi}{\partial r} = -C\left(1 + ikr\right) e^{i(\omega t - kr)}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = -C(ik)(ikr)e^{i(\omega t - kr)} = Ck^2 r e^{i(\omega t - kr)}$$

Therefore, the Laplacian is:

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = \frac{1}{r^2}\left(Ck^2 r e^{i(\omega t - kr)}\right) = \frac{Ck^2}{r} e^{i(\omega t - kr)}$$

$$\boxed{\nabla^2 \phi = \frac{Ck^2}{r} e^{i(\omega t - kr)} = k^2\,\frac{C}{r} e^{i(\omega t - kr)} = k^2 \phi}$$

The time derivative is:

$$\ddot{\phi} = \frac{\partial^2 \phi}{\partial t^2} = (i\omega)^2 \frac{C}{r} e^{i(\omega t - kr)} = -\omega^2 \frac{C}{r} e^{i(\omega t - kr)}$$

Plugging into the wave equation:

$$\frac{1}{\alpha^2}\ddot{\phi} = \nabla^2 \phi$$

$$\frac{1}{\alpha^2}\left(-\omega^2\frac{C}{r}e^{i(\omega t-kr)}\right)=\frac{Ck^2}{r}e^{i(\omega t-kr)}$$

Canceling common terms, we get the dispersion relation:

$$-\frac{\omega^2}{\alpha^2}=k^2 \quad \text{or} \quad k^2=\frac{\omega^2}{\alpha^2}$$

This confirms that k is constant, and our solution is valid. It is for this reason that solution in Part 1 above is wrong because k has an extra term which is imaginary.

Final Solution

The exact 3D spherical wave solution is:

$$\boxed{\phi(r,t)=A(r)e^{i(\omega t-kr)}=\frac{1}{r}e^{i(\omega t-kr)}}$$

Key Summary:

- ✓ **Geometrical spreading**: amplitude decreases as 1/r
- ✓ **Phase propagation**: $e^{i(\omega t-kr)}$ travels at speed α
- ✓ **Energy conservation**: $|\phi|^2\times r^2=\text{constant}$
- ✓ **Singularity at origin**: $\phi\to\infty$ as $r\to 0$ (point source)

Part 4: The Asymptotic 2D "Spherical" (Cylindrical) Wave Solution

In many applications (e.g., a long line source), waves spread out cylindrically in a 2D plane. The Laplacian for cylindrical coordinates with radial symmetry is:

$$\nabla^2\phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)$$

By analogy with the 3D case above, one might try a solution with a $1/\sqrt{r}$ amplitude decay, as the perimeter of a circle is $2\pi r$, suggesting intensity decays as $1/r$, and thus amplitude as $1/\sqrt{r}$.

$$\phi(r,t) = \frac{1}{\sqrt{r}} e^{i(\omega t - kr)}$$

Now, we will substitute this trial solution into the scalar wave equation to see if it works.

Computing the Derivatives

First, compute the spatial derivatives:

$$\frac{\partial \phi}{\partial r} = \left(-\frac{1}{2} r^{-3/2} - ikr^{-1/2}\right) e^{i(\omega t - kr)} = -\left(\frac{1}{2r} + ik\right) \frac{1}{\sqrt{r}} e^{i(\omega t - kr)}$$

Now compute $r \frac{\partial \phi}{\partial r}$:

$$r \frac{\partial \phi}{\partial r} = -\left(\frac{1}{2} + ikr\right) \frac{1}{\sqrt{r}} e^{i(\omega t - kr)}$$

Now compute the Laplacian term:

$$\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) = \frac{\partial}{\partial r}\left[-\left(\frac{1}{2} + ikr\right) r^{-1/2} e^{i(\omega t - kr)}\right]$$

Let $f(r) = -\left(\frac{1}{2} + ikr\right) e^{i(\omega t - kr)}$

Using the product rule:

$$f'(r) = -\left[(ik) r^{-1/2} + \left(\frac{1}{2} + ikr\right)\left(-\frac{1}{2} r^{-3/2} - ikr^{-1/2}\right)\right] e^{i(\omega t - kr)}$$

Let's compute this carefully:

- Derivative of first part: $\frac{d}{dr}[\frac{1}{2} + ikr] = ik$
- So first term: $-(ik) r^{\frac{-1}{2}} e^{i(\omega t - kr)}$
- Second term: $-\left(\frac{1}{2} + ikr\right) \times \frac{d}{dr}[r^{-1/2} e^{i(\omega t - kr)}]$
- $\frac{d}{dr}[r^{-1/2} e^{i(\omega t - kr)}] = \left(-\frac{1}{2} r^{-3/2} - ikr^{-1/2}\right) e^{i(\omega t - kr)}$

Putting it all together:

$$f'(r) = -ikr^{-1/2} e^{-ikr} - \left(\frac{1}{2} + ikr\right)\left(-\frac{1}{2} r^{-3/2} - ikr^{-1/2}\right) e^{i(\omega t - kr)}$$

Multiply out the second term:

$$\left(\frac{1}{2}+ikr\right)\left(-\frac{1}{2}r^{-3/2}-ikr^{-1/2}\right) = -\frac{1}{4}r^{-3/2}-\frac{ik}{2}r^{-1/2}-\frac{ik}{2}r^{-1/2}+k^2r^{1/2}$$
$$= -\frac{1}{4}r^{-3/2}-ikr^{-1/2}+k^2r^{1/2}$$

So:

$$\begin{aligned} f'(r) &= -ikr^{-1/2}e^{-ikr}-\left(-\frac{1}{4}r^{-3/2}-ikr^{-1/2}+k^2r^{1/2}\right)e^{i(\omega t-kr)} \\ &= \left(-ikr^{-1/2}+\frac{1}{4}r^{-3/2}+ikr^{-1/2}-k^2r^{1/2}\right)e^{i(\omega t-kr)} \\ &= \left(\frac{1}{4}r^{-3/2}-k^2r^{1/2}\right)e^{i(\omega t-kr)} \end{aligned}$$

Therefore:

$$\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \left(\frac{1}{4}r^{-3/2}-k^2r^{1/2}\right)e^{i(\omega t-kr)}$$

Now the full Laplacian:

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \frac{1}{r}\left(\frac{1}{4}r^{-3/2}-k^2r^{1/2}\right)e^{i(\omega t-kr)}$$

$$\nabla^2\phi = \left(\frac{1}{4r^{5/2}}-\frac{k^2}{r^{1/2}}\right)e^{i(\omega t-kr)} \quad \text{(Laplacian)}$$

$$\boxed{\nabla^2\phi = \left(\frac{1}{4r^2}-k^2\right)\frac{1}{\sqrt{r}}e^{i(\omega t-kr)} = \left(\frac{1}{4r^2}-k^2\right)\phi}$$

Now compute the time derivative:

$$\boxed{\ddot{\phi} = \frac{\partial^2\phi}{\partial t^2} = -\omega^2\frac{1}{\sqrt{r}}e^{i(\omega t-kr)}} \quad \text{(Time Derivative)}$$

Substituting into the Wave Equation

$$\frac{1}{\alpha^2}\ddot{\phi} = \nabla^2\phi$$

$$\frac{1}{\alpha^2}\left(-\omega^2\frac{1}{\sqrt{r}}e^{i(\omega t-kr)}\right)=\left(\frac{1}{4r^{5/2}}-\frac{k^2}{r^{1/2}}\right)e^{i(\omega t-kr)}$$

$$\frac{-\omega^2}{\alpha^2}\frac{1}{\sqrt{r}}e^{i(\omega t-kr)}=\left(\frac{1}{4r^2}-k^2\right)\frac{1}{\sqrt{r}}e^{i(\omega t-kr)}$$

Multiply both sides by $\sqrt{r}e^{-i(\omega t-kr)}$:

$$-\frac{\omega^2}{\alpha^2}=\frac{1}{4r^2}-k^2$$

Rearranging:

$$\boxed{k^2=\frac{\omega^2}{\alpha^2}+\frac{1}{4r^2}}\quad\text{(Dispersion Relation)}$$

This is the key result that shows why the trial solution above fails as an **exact** solution. k^2 should be equal to $\frac{\omega^2}{\alpha^2}$.

Physical Interpretation and Asymptotic Solution

The dispersion relation $k^2=\frac{\omega^2}{\alpha^2}+\frac{1}{4r^2}$ tells us:

1. **The wavenumber k depends on distance r** - this is unusual and problematic
2. **The wave is dispersive** - different frequency components travel at different speeds
3. **The waveform changes shape** as it propagates

However, in the **far field** ($r \gg \lambda$), the term $\frac{1}{4r^2}$ becomes negligible compared to $\frac{\omega^2}{\alpha^2}$ because:

- $k=\frac{2\pi}{\lambda}$, so $\frac{\omega^2}{\alpha^2}=k_0^2$ where k_0 is the constant wavenumber
- When $r>10\lambda$, $\frac{1}{4r^2}\ll k_0^2$

Therefore, in the **far field**:

$$k^2\approx\frac{\omega^2}{\alpha^2}\quad\Rightarrow\quad k\approx\frac{\omega}{\alpha}=k_0$$

Thus, in the **far field**, our trial solution becomes approximately valid:

$$\phi(r,t)\approx\frac{1}{\sqrt{r}}e^{i(\omega t-k_0r)}$$

More generally, for any waveform:

$$\boxed{\phi(r,t) \approx \frac{1}{\sqrt{r}} f(\omega t - kr)} \quad \text{works well!}$$

So, the trial solution above is an asymptotic solution of the wave equation in the **far field** in the 2D spherical coordinate.

The $1/\sqrt{r}$ amplitude decay represents **cylindrical geometrical spreading** - the wave energy spreads over cylinders with perimeter proportional to r, so intensity decays as $1/r$, and amplitude as $1/\sqrt{r}$.

This derivation shows why 2D wave propagation is fundamentally more complex than 3D propagation, requiring the far-field approximation for simple solutions.

Summary of Part 4

Aspect	**Mathematical Expression**	**Physical Meaning**
Exact Relation	$k^2 = \frac{\omega^2}{\alpha^2} + \frac{1}{4r^2}$	Wavenumber depends on position
Far Field Condition	$r \gg \lambda$	Distance much greater than wavelength
Asymptotic Relation	$k^2 \approx \frac{\omega^2}{\alpha^2}$	Constant wavenumber recovered
Asymptotic Solution	$\phi(r,t) \approx \frac{1}{\sqrt{r}} f(\omega t - k_0 r)$	Wave with cylindrical spreading

Note: The exact 2D solution involves **Hankel functions**, but $1/\sqrt{r}$ is excellent for $r \gg \lambda$.

General Summary

Dimension	**Spreading Factor**	**Exact Solution?**	**Physical Meaning**
3D	$1/r$	✓ Yes	Energy over sphere area $\propto r^2$
2D	$1/\sqrt{r}$	✓ Only asymptotic	Energy over circle $\propto r$
1D	1 (no decay)	✓ Yes	Plane wave, no spreading

Wave Type	**Mathematical Form**	**Intensity**	**Distance Dependence**
Plane (1D)	$Ae^{i(\omega t-kx)}$	$\|A\|^2$	**No decay**
Cylindrical (2D)	$\frac{A}{\sqrt{r}}e^{i(\omega t-kr)}$ (asymptotic, not exact)	$\frac{\|A\|^2}{r}$	$I \propto \frac{1}{r}$
Spherical (3D)	$\frac{A}{r}e^{i(\omega t-kr)}$	$\frac{\|A\|^2}{r^2}$	$I \propto \frac{1}{r^2}$