Acoustic Sensitivity Kernel

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Raypaths and Wavepaths

Wavepath and raypath are fundamental concepts in wave physics (especially seismology, acoustics, optics), and they are related but distinct.

Raypath

A *raypath* is an idealized, geometric path/line that represents the trajectory of seismic energy (or any wave energy) as if it traveled along a single, infinitesimally thin path.

Key characteristics

- Based on **ray theory** (or geometric optics/seismology), which assumes **high-frequency** waves.
- It assumes infinite frequency, meaning the wave has no wavelength (a pure mathematical line). This allows us to ignore wave effects like diffraction and interference.
- Treats wave propagation like light rays.
- Shows arrival direction, reflection, refraction, and bending due to velocity changes.
- Ignores finite-wavelength effects such as diffraction and scattering.
- **Governing Principle: Fermat's Principle (or the Principle of Least Time)**: The raypath between two points is the path that minimizes (or makes stationary) the travel time.

Use cases

- Travel-time tomography
- Reflection/refraction seismic interpretation

- First-arrival modeling

Wavepath

A *wavepath* is the volumetric region through which wave energy travels — not just a single line but a broader zone influenced by the finite wavelength (and thus, finite frequency) of the wave. A wavepath is a more physical, volumetric concept that represents the region of a medium actually influenced by a propagating wave. Energy doesn't just travel on the infinitesimally thin line (geometric raypath), but spreads out into this region.

Key characteristics

- Accounts for **finite-frequency effects**.
- Shows where seismic waves are actually sensitive to material properties.
- **Governing Principle**: Wave theory and scattering.
- Often represented/visualize as:
  - **Fresnel zones** (for transmitted waves), or
  - **Sensitivity kernels** (banana–doughnut kernels) in finite-frequency tomography.
- Energy spreads out around the raypath; the medium within the wavepath influences travel time and amplitude.

Use cases

- Finite-frequency seismic tomography
- Full-waveform inversion (FWI)
- Amplitude and waveform modeling

Raypath vs. Wavepath: Key Differences

Feature	**Raypath**	**Wavepath**
Nature	Single geometric line, **idealized**	Finite-volume region, **realistic**
Physics	Based on ray theory (optics)	Based on wave theory (scattering, diffraction)
Frequency assumption	High frequency (geometric optics)	Finite frequency (finite wavelength)
Accounts for diffraction/scattering?	No	Yes
Sensitivity to material changes	Only exactly on the ray	Region around ray (Fresnel zone / sensitivity kernel)
Use cases	Travel-time analysis	Finite-frequency tomography, FWI

Simple Analogy

- **Raypath**: The straight (or bent) line of a laser pointer.

- **Wavepath**: The beam of a flashlight — spreads out and interacts with a larger area.

A. Derivation Based on the Born Approximation

1 Frequency-Domain Helmholtz Equation

The acoustic frequency-domain wave equation is

$$-\omega^2\kappa^{-1}(\mathbf{x})P - \nabla \cdot (\rho^{-1}(\mathbf{x})\nabla P) = S(\mathbf{x}, \omega)$$

$$\left(\nabla^2 + \omega^2 m(\mathbf{x})\right) P(\mathbf{x}, \omega = -S(\mathbf{x}, \omega). \tag{1}$$

where:

- $\kappa(\mathbf{x})$ is the **bulk modulus**,
- $\rho(\mathbf{x})$ is the **density**,
- $S(\mathbf{x}, \omega)$ is the **source term**.
- $m(\mathbf{x}) = 1/c^2(\mathbf{x})$

For a point source at $\mathbf{s}$, we assume:

$$S(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{s}). \tag{2}$$

Let the model be decomposed as

$$m(\mathbf{x}) = m_0(\mathbf{x}) + \delta m(\mathbf{x}), \tag{3}$$

The unperturbed field P_0 satisfies:

$$\left[\nabla^2 + \omega^2 m_0\right] P_0(\mathbf{x}, \mathbf{s}) = -\delta(\mathbf{x} - \mathbf{s})$$

with solution:

$$P_0(\mathbf{x}, \mathbf{s}) = G_0(\mathbf{x} - \mathbf{s})$$

The perturbed field is:

$$P = P_0 + \delta P$$

2 Derivation of Perturbation Equation

Substitute $m = m_0 + \delta m$ and $P = P_0 + \delta P$ into the wave equation:

$$\left(\nabla^2 + \omega^2 m(\mathbf{x})\right) P(\mathbf{x}, \omega) = -S(\mathbf{x}, \omega)$$

$$\nabla^2(P_0 + \delta P) + \omega^2(m_0 + \delta m)(P_0 + \delta P) = -\delta(\mathbf{x} - \mathbf{s})$$

Separate terms:

$$[\nabla^2 P_0 + \omega^2 m_0 P_0] + [\nabla^2 \delta P + \omega^2 m_0 \delta P] + \omega^2 \delta m(P_0 + \delta P) = -\delta(\mathbf{x} - \mathbf{s})$$

The first bracket equals $-\delta(\mathbf{x} - \mathbf{s})$ by definition of P_0, canceling the RHS:

$$\cancel{-\delta(\mathbf{x} - \mathbf{s})} + [\nabla^2 \delta P + \omega^2 m_0 \delta P] + \omega^2 \delta m(P_0 + \delta P) = \cancel{-\delta(\mathbf{x} - \mathbf{s})}$$

$$\nabla^2 \delta P + \omega^2 m_0 \delta P + \omega^2 \delta m(P_0 + \delta P) = 0$$

Rearranged:

$$\boxed{-\left[\nabla^2 \delta P + \omega^2 m_0 \delta P\right] = \omega^2 \delta m(P_0 + \delta P)}$$

3 Born Approximation and Integral Solution (Lippmann-Schwinger)

Assuming δm is small, we can assume δP is small compared to P_0. So, we neglect the second-order term $\delta m\, \delta P$ (second order in perturbation):

$$-\left[\nabla^2 \delta P + \omega^2 m_0 \delta P\right] = \omega^2 \delta m P_0 + \underbrace{\omega^2 \delta m \delta P}_{\textbf{drop this}} \tag{4}$$

$$-\left[\nabla^2 \delta P + \omega^2 m_0 \delta P\right] \approx \omega^2 \delta m\, P_0$$

This can be written as:

$$\left[\nabla^2 + \omega^2 m_0\right] \delta P = -\omega^2 \delta m\, P_0 \tag{5}$$

Equation (5) is a Helmholtz equation for δP with source term $-\omega^2 \delta m\, P_0$. Using the Green's function G_0, where:

$$\left[\nabla^2 + \omega^2 m_0\right] G_0(\mathbf{r}, \mathbf{x}) = -\delta(\mathbf{r} - \mathbf{x})$$

with $G_0 = \frac{e^{i|\mathbf{r}-\mathbf{x}|}}{4\pi|\mathbf{r}-\mathbf{x}|}$

the solution is:

Using the convolution property:

$$\delta P(\mathbf{r}, \mathbf{s}) = \int_\Omega G_0(\mathbf{r}, \mathbf{x}) \delta(\mathbf{r} - \mathbf{x}) d\mathbf{x}. \tag{6}$$

$$\delta P(\mathbf{r}, \mathbf{s}) = \int_\Omega G_0(\mathbf{r}, \mathbf{x}) \cdot \omega^2 \delta m\, P_0 \tag{7}$$

$$\boxed{\delta P(\mathbf{r}, \mathbf{s}) \approx \int_\Omega \omega^2 G_0(\mathbf{r}, \mathbf{x}) \delta m(\mathbf{x}) P_0(\mathbf{x}, \mathbf{s})\, d\mathbf{x}} \tag{8}$$

$P_0(\mathbf{x}, \omega)$ represents the background or unperturbed wavefield from the source point $\mathbf{s}$ to the scatterer at $\mathbf{x}$ which can be represented as $G_0(\mathbf{x}, \omega, \mathbf{s})$

If we had not dropped the second order perturbation term in equation 4, then equation 8 would be like:

$$\boxed{\delta P(\mathbf{r}, \mathbf{s}) = \int_\Omega G_0(\mathbf{r}, \mathbf{x}) \left[\omega^2 \delta m(\mathbf{x}) P_0(\mathbf{x}, \mathbf{s}) + \omega^2 \delta m(\mathbf{x}) \delta P\right]\, d\mathbf{x}} \tag{9}$$

Thus, the final integral representation is:

$$\boxed{\delta P(\mathbf{r}, \mathbf{s}) \approx \int_\Omega \omega^2 G_0(\mathbf{r}, \mathbf{x}) \delta m(\mathbf{x}) P_0(\mathbf{x}, \mathbf{s})\, d\mathbf{x}} \tag{10}$$

This is the exact Lippmann–Schwinger form for the scattered field. This represents the scattered wavefield in terms of Green's function and the perturbation in medium properties. The scattered wave measured at $\mathbf{r}$ is the superposition (integral) of contributions from all point scatterers $\delta m(\mathbf{x})$, each one re-radiating (weighted by the incident amplitude at $\mathbf{x}$ and the Green's function from $\mathbf{x}$ to $\mathbf{r}$).

The integral representation of the total wavefield,we have:

$$P(\mathbf{r}) = P_0(\mathbf{r}) + \delta P(\mathbf{r})$$

$$P(\mathbf{r}) = P_0(\mathbf{r}) + \int_\Omega G_0(\mathbf{r},\mathbf{x})\,\omega^2 \delta m(\mathbf{x})\,P_0(\mathbf{x})\,d\mathbf{x}, \tag{11}$$

where the background (unperturbed) field is

$$\left(\nabla^2 + \omega^2 m_0(\mathbf{x})\right) P_0(\mathbf{x},\omega) = -S(\mathbf{x},\omega)$$

$$P_0(\mathbf{r}) = \int G_0(\mathbf{r},\mathbf{x}) S(\mathbf{x})\,d\mathbf{x}, \tag{12}$$

and for a point source at $\mathbf{s}$,

$$S(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{s}), \qquad P_0(\mathbf{r}) = G_0(\mathbf{r},\mathbf{s}). \tag{13}$$

Rearranging the integral equation for the scattered field $\delta P(\mathbf{r}) = P(\mathbf{r}) - P_0(\mathbf{r})$ gives equation 10.

4 Wavepath Sensitivity Kernel $K_m P$

Define the kernel (the "wavepath" sensitivity kernel) as the factor multiplying $\delta m(\mathbf{x})$ in the Born integral

$$K_m P(\mathbf{r},\mathbf{x}) := \omega^2\, G_0(\mathbf{r},\mathbf{x})\, P_0(\mathbf{x},\mathbf{s})$$

This is the Fréchet (Born) kernel.

So the Born linear mapping is

$$\delta P(\mathbf{r},\mathbf{s}) = \int_\Omega K_m P(\mathbf{r},\mathbf{x})\,\delta m(\mathbf{x})\,d\mathbf{x}$$

Since $P_0(\mathbf{x},\mathbf{s}) = G_0(\mathbf{x}-\mathbf{s})$ (for a unit impulsive source at $\mathbf{s}$):

$$K_m P(\mathbf{r},\mathbf{x}) := \omega^2\, G_0(\mathbf{r},\mathbf{x})\, \underbrace{P_0(\mathbf{x},\mathbf{s})}_{\sim G_0(\mathbf{x},\mathbf{s})}$$

$$K_m P(\mathbf{r},\mathbf{x}) = \omega^2\, G_0(\mathbf{r},\mathbf{x})\, G_0(\mathbf{x},\mathbf{s})$$

Substituting the Green's function expression:

In 3D homogeneous space:

$$G_0(\mathbf{r}, \mathbf{x}) = \frac{1}{4\pi|\mathbf{r} - \mathbf{x}|} e^{ik_0|\mathbf{r}-\mathbf{x}|}, \qquad k_0 = \frac{\omega}{v_0}. \tag{14}$$

For a point source at $\mathbf{s}$,

$$P_0(\mathbf{x}, \mathbf{s}) = G_0(\mathbf{x}, \mathbf{s}) = \frac{1}{4\pi|\mathbf{x} - \mathbf{s}|} e^{ik_0|\mathbf{x}-\mathbf{s}|}. \tag{15}$$

Thus:

$$K_m P(\mathbf{r}, \mathbf{x}) = \omega^2 \frac{1}{(4\pi)^2} \frac{e^{ik_0(|\mathbf{x}-\mathbf{r}|+|\mathbf{x}-\mathbf{s}|)}}{|\mathbf{x} - \mathbf{r}|\,|\mathbf{x} - \mathbf{s}|} \tag{16}$$

$$\boxed{K_m P(\mathbf{r}, \mathbf{x}) = \left(\frac{\omega}{4\pi}\right)^2 \frac{1}{|\mathbf{x} - \mathbf{r}||\mathbf{x} - \mathbf{s}|} e^{ik_0(|\mathbf{x}-\mathbf{r}|+|\mathbf{x}-\mathbf{s}|)}} \tag{17}$$

$K_m P$ is often called a sensitivity kernel (or Fréchet kernel). It is *not* exactly the gradient of a simple multivariate function but it is the kernel of the linearized forward map from δm to the data δP. The kernel contains a geometric amplitude factor $1/(|\mathbf{r} - \mathbf{x}|, |\mathbf{x} - \mathbf{s}|)$ and a phase $e^{ik_0(|r-x|+|x-s|)}$. The phase depends on the total path length from source $\mathbf{s}$ to scatterer $\mathbf{x}$ to receiver $\mathbf{r}$.

5 Fresnel Zones as Confocal Ellipses (equal-phase loci)

The phase of the kernel is constant when

$$L(\mathbf{x}) = |\mathbf{x} - \mathbf{s}| + |\mathbf{r} - \mathbf{x}| = \text{constant}. \tag{18}$$

Surfaces (in 2-D cross sections: curves) of constant L are confocal ellipses with foci at $\mathbf{s}$ and $\mathbf{r}$. This is the definition of a confocal ellipse.

Definition (ellipse by foci): an ellipse is the locus of points $\mathbf{x} = (x, y)$ such that the sum of distances to two fixed foci equals a constant (2a):

$$|\mathbf{x} - \mathbf{s}| + |\mathbf{x} - \mathbf{r}| = 2a.$$

If the foci lie at $(\pm c, 0)$ (i.e. separated by (2c)) and the ellipse is centered at the origin, the standard ellipse equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad b^2 = a^2 - c^2.$$

Let the foci be at $\mathbf{s} = (-c, 0)$ and $\mathbf{r} = (c, 0)$. Points satisfying

$$|\mathbf{x} - \mathbf{s}| + |\mathbf{x} - \mathbf{r}| = 2a \tag{19}$$

form an ellipse with:

$$\text{semi-major axis:} \quad a, \tag{20}$$

$$\text{semi-minor axis:} \quad b = \sqrt{a^2 - c^2}, \tag{21}$$

$$\text{focal distance:} \quad 2c. \tag{22}$$

c is the distance from center to each focus, with $0 \leq c < a$. $2a$ is the constant total path length (sum of distances to foci)

Thus the ellipse equation is

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.} \tag{23}$$

<u>Fresnel Zones</u>:

When the phase $k_0 L(\mathbf{x})$ differs by integer multiples of 2π, i.e.

Successive Fresnel zones correspond to

$$k_0 L(\mathbf{x}) = \text{constant} + 2\pi n$$

$$k_0 \left(|\mathbf{x} - \mathbf{s}| + |\mathbf{r} - \mathbf{x}|\right) = \text{constant} + 2\pi n. \tag{24}$$

the points $\mathbf{x}$ lie on confocal ellipses with discrete values of a. These annular elliptical regions between successive ellipses are the Fresnel zones: within the first Fresnel zone the scattered contributions add mostly in phase at the receiver; outside they tend to cancel more. These zones show constructive and destructive interference around the geometric ray path.

Thus the equal-phase curves of the kernel are confocal ellipses (2-D slices) or confocal ellipsoids (3-D) with foci at source and receiver. That is why the kernel's significant energy is concentrated in a "banana" or "peanut" shaped region around the geometrical ray: the first few Fresnel zones around the ray contain most of the coherent sensitivity.

B. Derivation Based on the Rytov Approximation

We can write the acoustic wave equation above again as:

$$-\left(\nabla^2 + \omega^2 m(\mathbf{x})\right) P(\mathbf{x}, \omega) = S(\mathbf{x}, \omega).$$

$$\left[\nabla^2 + \omega^2 m(\mathbf{x})\right] P(\mathbf{x}) = 0$$

where $m(\mathbf{x}) = 1/c^2(\mathbf{x})$ is the slowness squared, ω is angular frequency, and $c(\mathbf{x})$ is wave velocity.

Rytov's Complex Phase Representation

Rytov's key insight is to represent the wavefield as:

$$P(\mathbf{x}) = e^{\phi(\mathbf{x}, \omega)}$$

where $\phi(\mathbf{x}) = \phi_0(\mathbf{x}) + \delta\phi(\mathbf{x})$ is a **complex phase function** containing both amplitude and phase information.

Wave Equation in Complex Phase Form

First Derivative

$$\boxed{\nabla P = \nabla e^{\phi} = (\nabla\phi)e^{\phi} = (\nabla\phi)P}$$

Second Derivative

Using the product rule:

$$\nabla^2 P = \nabla \cdot (\nabla P) = \nabla \cdot [(\nabla\phi)P]$$

Since $P = e^{\phi}$:

$$\nabla^2 P = \nabla \cdot [(\nabla\phi)e^{\phi}] = [\nabla^2\phi + (\nabla\phi)^2]e^{\phi}$$

Thus:

$$\boxed{\nabla^2 P = [\nabla^2\phi + (\nabla\phi)^2]P}$$

Substitute the derivatives into the wave equation

$$\nabla^2 P + \omega^2 m P = 0$$

Substituting:

$$[\nabla^2\phi + (\nabla\phi)^2]P + \omega^2 m P = 0$$

Since $P \neq 0$, divide by P:

$$\boxed{\nabla^2\phi + (\nabla\phi)^2 + \omega^2 m = 0}$$

This is the **exact** wave equation in Rytov's complex phase form.

Derivation of Perturbation Equation

Let:

$$m(\mathbf{x}) = m_0 + \delta m(\mathbf{x})$$

$$\phi(\mathbf{x}) = \phi_0(\mathbf{x}) + \delta\phi(\mathbf{x})$$

where ϕ_0 is the background phase and $\delta\phi$ is the phase perturbation.

For the background (unperturbed) medium (m_0 constant):

$$\nabla^2\phi_0 + (\nabla\phi_0)^2 + \omega^2 m_0 = 0 \tag{25}$$

For the perturbed equation:

Substitute $m = m_0 + \delta m$ and $\phi = \phi_0 + \delta\phi$ into (2):

$$\nabla^2(\phi_0 + \delta\phi) + [\nabla(\phi_0 + \delta\phi)]^2 + \omega^2(m_0 + \delta m) = 0$$

Expand the gradient squared term:

$$[\nabla(\phi_0 + \delta\phi)]^2 = (\nabla\phi_0)^2 + 2\nabla\phi_0 \cdot \nabla\delta\phi + (\nabla\delta\phi)^2$$

So the equation becomes:

$$\nabla^2\phi_0 + \nabla^2\delta\phi + (\nabla\phi_0)^2 + 2\nabla\phi_0 \cdot \nabla\delta\phi$$
$$+ (\nabla\delta\phi)^2 + \omega^2 m_0 + \omega^2 \delta m = 0 \tag{26}$$

Subtract the background equation (25) from (26):

$$[\nabla^2\phi_0 + (\nabla\phi_0)^2 + \omega^2 m_0] + \nabla^2\delta\phi + 2\nabla\phi_0 \cdot \nabla\delta\phi + (\nabla\delta\phi)^2 + \omega^2\delta m = 0$$

The terms in brackets sum to zero (from equation 25), so:

$$\boxed{\nabla^2\delta\phi + 2\nabla\phi_0 \cdot \nabla\delta\phi + (\nabla\delta\phi)^2 + \omega^2\delta m = 0} \tag{27}$$

This is the **exact perturbation equation** for Rytov's method.

Linearization

We want to obtain a linear equation for $\delta\phi$. Notice the problematic term $(\nabla\delta\phi)^2$ which is **nonlinear**.

Rearrange the perturbation equation:

$$-[\omega^2\delta m + 2\nabla\phi_0 \cdot \nabla\delta\phi + (\nabla\delta\phi)^2 + \nabla^2\delta\phi] = 0 \tag{28}$$

We would like something of the form:

$$\boxed{-[\omega^2 m_0 + \nabla^2](P_0\delta\phi) = \text{RHS}(\delta m, P_0)}$$

Using $P_0 = e^{\phi_0}$, let's compute $\nabla^2(P_0\delta\phi)$:

First Derivative

$$\nabla(P_0\delta\phi) = (\nabla P_0)\delta\phi + P_0\nabla\delta\phi$$

Since $P_0 = e^{\phi_0}$, $\nabla P_0 = (\nabla\phi_0)P_0$:

$$\nabla(P_0\delta\phi) = P_0[(\nabla\phi_0)\delta\phi + \nabla\delta\phi]$$

Second Derivative

$$\nabla^2(P_0\delta\phi) = \nabla \cdot [P_0(\nabla\phi_0\delta\phi + \nabla\delta\phi)]$$

Expand:

$$\nabla^2(P_0\delta\phi) = (\nabla P_0) \cdot (\nabla\phi_0\delta\phi + \nabla\delta\phi) + P_0\nabla \cdot (\nabla\phi_0\delta\phi + \nabla\delta\phi)$$

First term: $\nabla P_0 = (\nabla\phi_0)P_0$, so:

$$(\nabla P_0) \cdot (\nabla\phi_0\delta\phi + \nabla\delta\phi) = P_0[(\nabla\phi_0)^2\delta\phi + \nabla\phi_0 \cdot \nabla\delta\phi]$$

Second term:

$$\nabla \cdot (\nabla\phi_0\delta\phi + \nabla\delta\phi) = \nabla^2\phi_0\delta\phi + \nabla\phi_0 \cdot \nabla\delta\phi + \nabla^2\delta\phi$$

So:

$$\begin{aligned}\nabla^2(P_0\delta\phi) = P_0[&(\nabla\phi_0)^2\delta\phi + \nabla\phi_0 \cdot \nabla\delta\phi \\ &+ \nabla^2\phi_0\delta\phi + \nabla\phi_0 \cdot \nabla\delta\phi + \nabla^2\delta\phi]\end{aligned}$$

Combine terms:

$$\boxed{\nabla^2(P_0\delta\phi) = P_0[(\nabla\phi_0)^2\delta\phi + \nabla^2\phi_0\delta\phi + 2\nabla\phi_0 \cdot \nabla\delta\phi + \nabla^2\delta\phi]} \tag{29}$$

From (27), we have:

$$\nabla^2\delta\phi + 2\nabla\phi_0 \cdot \nabla\delta\phi = -[(\nabla\delta\phi)^2 + \omega^2\delta m]$$

Multiply by P_0:

$$P_0[\nabla^2\delta\phi + 2\nabla\phi_0 \cdot \nabla\delta\phi] = -P_0[(\nabla\delta\phi)^2 + \omega^2\delta m] \tag{30}$$

From (29), note that:

$$P_0[\nabla^2\delta\phi + 2\nabla\phi_0 \cdot \nabla\delta\phi] = \nabla^2(P_0\delta\phi) - P_0[(\nabla\phi_0)^2\delta\phi + \nabla^2\phi_0\delta\phi] \tag{31}$$

Using Background Equation for ϕ_0

From the background equation (25):

$$\nabla^2\phi_0 + (\nabla\phi_0)^2 + \omega^2 m_0 = 0$$

Multiply by $P_0\delta\phi$:

$$P_0[(\nabla\phi_0)^2 + \nabla^2\phi_0]\delta\phi = -\omega^2 m_0 P_0\delta\phi \tag{32}$$

Substitute (31) into (30):

$$\nabla^2(P_0\delta\phi) - P_0[(\nabla\phi_0)^2 + \nabla^2\phi_0]\delta\phi = -P_0[(\nabla\delta\phi)^2 + \omega^2\delta m]$$

Now use (32) to replace $P_0[(\nabla\phi_0)^2 + \nabla^2\phi_0]\delta\phi$:

$$\nabla^2(P_0\delta\phi) + \omega^2 m_0 P_0\delta\phi = -P_0[(\nabla\delta\phi)^2 + \omega^2\delta m]$$

Rearrange:

$$\boxed{-[\omega^2 m_0 + \nabla^2](P_0\delta\phi) = P_0[\omega^2\delta m + \underbrace{(\nabla\delta\phi)^2}_{\text{drop this}}]} \tag{33}$$

This is the **exact Rytov equation**.

The Rytov approximation assumes:

$$(\nabla\delta\phi)^2 \ll \omega^2\delta m$$

This is valid for **smooth perturbations** (disk diffractors) where $\delta\phi$ varies slowly.

With this approximation:

$$\boxed{-[\omega^2 m_0 + \nabla^2](P_0\delta\phi) \approx \omega^2\delta m P_0} \tag{34}$$

Comparison with Born Approximation

Born Form

Recall Born approximation for δP:

$$\boxed{-[\omega^2 m_0 + \nabla^2]\delta P \approx \omega^2 \delta m P_0}$$

Rytov Form

Rytov approximation:

$$\boxed{-[\omega^2 m_0 + \nabla^2](P_0 \delta\phi) \approx \omega^2 \delta m P_0}$$

Connection

For weak scattering ($\delta\phi \ll 1$):

$$\boxed{P = e^{\phi_0 + \delta\phi} = e^{\phi_0} e^{\delta\phi} \approx P_0(1 + \delta\phi)}$$

So:

$$\delta P = P - P_0 \approx P_0 \delta\phi$$

Thus in the weak-scattering limit, Rytov reduces to Born.

Sensitivity Kernel for Rytov

Equation (34) is:

$$[\nabla^2 + \omega^2 m_0](P_0 \delta\phi) = -\omega^2 \delta m P_0$$

This is identical in form to the Born equation, so we can solve using Green's functions:

$$P_0(\mathbf{r})\delta\phi(\mathbf{r}) = \int \omega^2 G_0(\mathbf{r}, \mathbf{x}) \delta m(\mathbf{x}) P_0(\mathbf{x}) d\mathbf{x}$$

Thus:

$$\delta\phi(\mathbf{r}) = \int \omega^2 \frac{G_0(\mathbf{r}, \mathbf{x})}{P_0(\mathbf{r})} \delta m(\mathbf{x}) P_0(\mathbf{x}) d\mathbf{x}$$

Sensitivity Kernel Definition

The sensitivity kernel for $\delta\phi$ is:

$$\boxed{K_m^{\phi}(\mathbf{r},\mathbf{x}) := \omega^2 \frac{G_0(\mathbf{r},\mathbf{x})P_0(\mathbf{x})}{P_0(\mathbf{r})}}$$

Explicit Form in 3D Homogeneous Medium

For a point source at $\mathbf{s}$:

$$P_0(\mathbf{x}) = G_0(\mathbf{x},\mathbf{s}) = \frac{e^{ik_0|\mathbf{x}-\mathbf{s}|}}{4\pi|\mathbf{x}-\mathbf{s}|}$$

$$G_0(\mathbf{r},\mathbf{x}) = \frac{e^{ik_0|\mathbf{r}-\mathbf{x}|}}{4\pi|\mathbf{r}-\mathbf{x}|}$$

$$P_0(\mathbf{r}) = G_0(\mathbf{r},\mathbf{s}) = \frac{e^{ik_0|\mathbf{r}-\mathbf{s}|}}{4\pi|\mathbf{r}-\mathbf{s}|}$$

Substituting:

$$K_m^{\phi}(\mathbf{r},\mathbf{x}) = \omega^2 \cdot \frac{\frac{e^{ik_0|\mathbf{r}-\mathbf{x}|}}{4\pi|\mathbf{r}-\mathbf{x}|} \cdot \frac{e^{ik_0|\mathbf{x}-\mathbf{s}|}}{4\pi|\mathbf{x}-\mathbf{s}|}}{\frac{e^{ik_0|\mathbf{r}-\mathbf{s}|}}{4\pi|\mathbf{r}-\mathbf{s}|}}$$

Simplify:

$$\boxed{K_m^{\phi}(\mathbf{r},\mathbf{x}) = \frac{\omega^2}{4\pi} \cdot \frac{e^{ik_0(|\mathbf{r}-\mathbf{x}|+|\mathbf{x}-\mathbf{s}|-|\mathbf{r}-\mathbf{s}|)}}{|\mathbf{r}-\mathbf{x}||\mathbf{x}-\mathbf{s}|/|\mathbf{r}-\mathbf{s}|}}$$

Physical Interpretation

Phase Factors Comparison

- **Born kernel**: $e^{ik_0(|\mathbf{x}-\mathbf{r}|+|\mathbf{x}-\mathbf{s}|)}$
- **Rytov kernel**: $e^{ik_0(|\mathbf{x}-\mathbf{r}|+|\mathbf{x}-\mathbf{s}|-|\mathbf{r}-\mathbf{s}|)}$

The Rytov kernel has **phase subtracted by the direct path** $|\mathbf{r}-\mathbf{s}|$, which represents **phase delay relative to unperturbed wave**.

Amplitude Factors Comparison

- **Born**: $\dfrac{1}{|\mathbf{x}-\mathbf{r}||\mathbf{x}-\mathbf{s}|}$
- **Rytov**: $\dfrac{1}{|\mathbf{x}-\mathbf{r}||\mathbf{x}-\mathbf{s}|/|\mathbf{r}-\mathbf{s}|}$

The Rytov kernel is **normalized by the source-receiver distance**.

When to Use Each Approximation

- **Born Approximation**: Point scatterers, sharp boundaries, strong perturbations
- **Rytov Approximation**: Smooth perturbations, disk diffractors, extended scatterers
- **Weak scattering limit**: Both approximations converge

Key Summary of Equations

1. Exact wave equation in phase form:

$$\nabla^2\phi + (\nabla\phi)^2 + \omega^2 m = 0$$

2. Exact perturbation equation:

$$\nabla^2\delta\phi + 2\nabla\phi_0 \cdot \nabla\delta\phi + (\nabla\delta\phi)^2 + \omega^2\delta m = 0$$

3. Exact Rytov equation:

$$-[\omega^2 m_0 + \nabla^2](P_0\delta\phi) = P_0[\omega^2\delta m + (\nabla\delta\phi)^2]$$

4. Rytov approximation (assuming $(\nabla\delta\phi)^2 \ll \omega^2\delta m$):

$$-[\omega^2 m_0 + \nabla^2](P_0\delta\phi) \approx \omega^2\delta m P_0$$

5. Rytov sensitivity kernel:

$$K_m^\phi(\mathbf{r},\mathbf{x}) = \frac{\omega^2}{4\pi} \cdot \frac{e^{ik_0(|\mathbf{r}-\mathbf{x}|+|\mathbf{x}-\mathbf{s}|-|\mathbf{r}-\mathbf{s}|)}}{|\mathbf{r}-\mathbf{x}||\mathbf{x}-\mathbf{s}|/|\mathbf{r}-\mathbf{s}|}$$

Conclusion

The Rytov approximation provides an alternative linearization scheme to the Born approximation, particularly well-suited for **smooth, extended scatterers** where phase perturbations accumulate gradually along the wave path. Unlike Born which linearizes the wavefield directly, Rytov linearizes the complex phase, making it more robust for certain types of smooth perturbations while maintaining the same mathematical structure for inversion.

C. Derivation Based on the Ray Approximation

The ray approximation is a high-frequency limit ($\omega \to \infty$) of wave propagation where waves can be treated as traveling along geometric rays. This is also known as **geometric optics** in optics or **geometric seismology** in seismology.

Travel Time Definition

For a ray path Γ from point A to point B, the travel time T is given by:

$$T = \int_A^B \frac{ds}{c(\mathbf{x})}$$

where ds is the arc length element along the ray, and $c(\mathbf{x})$ is the wave velocity.

Since $m(\mathbf{x}) = 1/c^2(\mathbf{x})$ is the slowness squared, we can write:

$$T = \int_A^B \sqrt{m(\mathbf{x})}\, ds$$

Ray Path Representation Using Dirac Delta

Integral Representation

We can represent the travel time as an integral over the entire domain Ω using the Dirac delta function:

$$T = \int_\Omega \sqrt{m(\mathbf{x})}\, \delta(\mathbf{x} - \mathbf{x}_{\text{ray}})\, d\mathbf{x}$$

where $\mathbf{x}_{\text{ray}}(s)$ is the position along the ray path parameterized by arc length s.

The delta function $\delta(\mathbf{x} - \mathbf{x}_{\text{ray}})$ is a **functional** that picks out the ray path in the domain. More precisely, we should write:

$$T = \int_A^B \sqrt{m(\mathbf{x}_{\text{ray}}(s))}\, ds = \int_\Omega \sqrt{m(\mathbf{x})}\delta_L(\mathbf{x} - \mathbf{x}_{\text{ray}})\, d\mathbf{x}$$

where δ_L is a line delta function concentrated on the ray path.

Perturbation Analysis

Background and Perturbed Quantities

Let:

- Background velocity: $c_0(\mathbf{x})$, with $m_0(\mathbf{x}) = 1/c_0^2(\mathbf{x})$

- Perturbed velocity: $c(\mathbf{x}) = c_0(\mathbf{x}) + \delta c(\mathbf{x})$, with $m(\mathbf{x}) = m_0(\mathbf{x}) + \delta m(\mathbf{x})$
- Background ray path: $\mathbf{x}_{\text{ray}}^{(0)}(s)$ (ray in the background medium)
- Perturbed ray path: $\mathbf{x}_{\text{ray}}(s)$ (ray in the perturbed medium)

The travel times are:

$$T_0 = \int_A^B \sqrt{m_0(\mathbf{x}_{\text{ray}}^{(0)}(s))}\, ds$$

$$T = \int_A^B \sqrt{m(\mathbf{x}_{\text{ray}}(s))}\, ds$$

Using Integral Representation

In delta function notation:

$$T_0 = \int_\Omega \sqrt{m_0(\mathbf{x})}\, \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})\, d\mathbf{x}$$

$$T = \int_\Omega \sqrt{m(\mathbf{x})}\, \delta(\mathbf{x} - \mathbf{x}_{\text{ray}})\, d\mathbf{x}$$

First-Order Perturbation Analysis

Taylor Expansion of Square Root

For small perturbations $\delta m \ll m_0$:

$$\sqrt{m_0 + \delta m} = \sqrt{m_0}\sqrt{1 + \frac{\delta m}{m_0}} = \sqrt{m_0}\left(1 + \frac{1}{2}\frac{\delta m}{m_0} - \frac{1}{8}\left(\frac{\delta m}{m_0}\right)^2 + \cdots\right)$$

Keeping only first-order terms:

$$\sqrt{m_0 + \delta m} \approx \sqrt{m_0} + \frac{1}{2}\frac{\delta m}{\sqrt{m_0}}$$

Since $\sqrt{m_0} = 1/c_0$, we can also write:

$$\sqrt{m_0 + \delta m} - \sqrt{m_0} \approx \frac{1}{2}\frac{\delta m}{\sqrt{m_0}} = -\frac{1}{2}c_0 \delta m$$

because $\delta m = -\frac{2}{c_0^3}\delta c$ (from $m = 1/c^2$).

Perturbation in Travel Time

The travel time perturbation is:

$$\delta T = T - T_0 = \int_\Omega \left[\sqrt{m(\mathbf{x})}\, \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}) - \sqrt{m_0(\mathbf{x})}\, \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)}) \right] d\mathbf{x} \tag{35}$$

The Born Approximation for Ray Theory

Key Simplification

In the **ray Born approximation** or **Fermat's principle approximation**, we make a crucial simplification:

$$\delta(\mathbf{x} - \mathbf{x}_{\text{ray}}) \approx \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})$$

This means we **ignore the change in ray path geometry** due to the velocity perturbation. This is valid when:

1. **High frequency limit** ($\omega \to \infty$): The ray is well-defined and doesn't bend much
2. **Small perturbations**: $\delta c / c_0 \ll 1$
3. **Smooth perturbations**: The velocity field varies slowly relative to wavelength

Physically, this is **Fermat's principle**: to first order, the travel time is stationary with respect to small changes in ray path.

Applying the Approximation

With this approximation, equation (35) becomes:

$$\boxed{\delta T \approx \int_\Omega \left[\sqrt{m(\mathbf{x})} - \sqrt{m_0(\mathbf{x})} \right] \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})\, d\mathbf{x}}$$

Detailed Derivation Step by Step

Start with Exact Perturbation

$$\delta T = \int_\Omega \sqrt{m(\mathbf{x})}\, \delta(\mathbf{x} - \mathbf{x}_{\text{ray}})\, d\mathbf{x} - \int_\Omega \sqrt{m_0(\mathbf{x})}\, \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})\, d\mathbf{x}$$

Apply Ray Path Approximation

$$\delta(\mathbf{x} - \mathbf{x}_{\text{ray}}) \approx \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)}) + \text{higher order terms}$$

Keeping only zeroth order in ray path perturbation:

$$\delta T \approx \int_\Omega \left[\sqrt{m(\mathbf{x})} - \sqrt{m_0(\mathbf{x})}\right] \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})\, d\mathbf{x}$$

Expand Square Root

$$\sqrt{m(\mathbf{x})} - \sqrt{m_0(\mathbf{x})} = \sqrt{m_0(\mathbf{x}) + \delta m(\mathbf{x})} - \sqrt{m_0(\mathbf{x})}$$

Taylor expansion:

$$\sqrt{m_0 + \delta m} - \sqrt{m_0} = \frac{1}{2}\frac{\delta m}{\sqrt{m_0}} - \frac{1}{8}\frac{(\delta m)^2}{m_0^{3/2}} + \cdots$$

Keep first order only:

$$\sqrt{m_0 + \delta m} - \sqrt{m_0} \approx \frac{1}{2}\frac{\delta m}{\sqrt{m_0}}$$

Substitute

$$\delta T \approx \int_\Omega \frac{1}{2}\frac{\delta m(\mathbf{x})}{\sqrt{m_0(\mathbf{x})}}\, \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})\, d\mathbf{x}$$

Express in Terms of Velocity

Since $\sqrt{m_0} = 1/c_0$, we have:

$$\frac{1}{\sqrt{m_0}} = c_0$$

Also, from $m = 1/c^2$, we get:

$$\delta m = m - m_0 = \frac{1}{c^2} - \frac{1}{c_0^2}$$

For small perturbations $c = c_0 + \delta c$:

$$\frac{1}{c^2} = \frac{1}{(c_0 + \delta c)^2} = \frac{1}{c_0^2(1 + \delta c/c_0)^2} = \frac{1}{c_0^2}\left(1 - 2\frac{\delta c}{c_0} + 3\frac{\delta c^2}{c_0^2} - \cdots\right)$$

So:

$$\delta m = \frac{1}{c^2} - \frac{1}{c_0^2} = -\frac{2}{c_0^3}\delta c + \frac{3}{c_0^4}\delta c^2 + \cdots$$

$$\delta m \approx -\frac{2}{c_0^3}\delta c$$

But it's cleaner to work with m directly. Using $\frac{1}{2}\frac{\delta m}{\sqrt{m_0}} = \frac{1}{2}c_0\delta m$:

Thus:

$$\boxed{\delta T \approx \int_\Omega \frac{1}{2}c_0(\mathbf{x})\delta m(\mathbf{x})\,\delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})\,d\mathbf{x}} \tag{36}$$

Sign Convention Analysis

Relationship Between δc and δm

From $m = 1/c^2$:

$$\delta m = m - m_0 = \frac{1}{c^2} - \frac{1}{c_0^2} \approx -\frac{2}{c_0^3}\delta c$$

Physical Consistency Check

Consider a velocity increase ($\delta c > 0$):

- Then $\delta m < 0$ (slowness squared decreases)
- Travel time should decrease ($\delta T < 0$) since waves travel faster

In equation (5):

- If $\delta c > 0$, then $\delta m < 0$
- So $\frac{1}{2}c_0\delta m < 0$
- Therefore $\delta T < 0$ (travel time decreases) ✓

Sensitivity Kernel Definition

The sensitivity kernel $K_m^T(\mathbf{x})$ is defined such that:

$$\delta T = \int_\Omega K_m^T(\mathbf{x})\delta m(\mathbf{x})\,d\mathbf{x}$$

From equation (36) (with corrected sign):

$$\delta T \approx \int_\Omega \frac{1}{2}c_0(\mathbf{x})\delta m(\mathbf{x})\,\delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})\,d\mathbf{x}$$

Therefore:

$$\boxed{K_m^T(\mathbf{x}) = \frac{1}{2}c_0(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})} \tag{37}$$

Alternative Derivation Using Fermat's Principle

Functional Derivative Approach

Travel time functional:

$$T[m] = \int_A^B \sqrt{m(\mathbf{x}(s))}\, ds$$

where $\mathbf{x}(s)$ is the ray path, which itself depends on m.

By Fermat's principle, the ray path minimizes T. For a first-order perturbation:

$$\delta T = \delta T_{\text{direct}} + \delta T_{\text{ray}}$$

where:

- δT_{direct} comes from changing m along the fixed path
- δT_{ray} comes from changing the path

But by Fermat's principle, to first order $\delta T_{\text{ray}} = 0$ (stationarity condition). So:

$$\delta T \approx \int_A^B \frac{1}{2\sqrt{m_0}} \delta m\, ds = \int_\Omega \frac{1}{2\sqrt{m_0(\mathbf{x})}} \delta m(\mathbf{x})\, \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})\, d\mathbf{x}$$

Since $\sqrt{m_0} = 1/c_0$:

$$\boxed{\delta T \approx \int_\Omega \frac{1}{2} c_0(\mathbf{x}) \delta m(\mathbf{x})\, \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})\, d\mathbf{x}}$$

Physical Interpretation

Kernel Structure

The ray sensitivity kernel is:

$$\boxed{K_m^T(\mathbf{x}) = \frac{1}{2} c_0(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})}$$

This is **infinitely thin** - it's only non-zero exactly on the geometric ray. This is the **high-frequency limit** where waves have no width.

Comparison with Finite-Frequency Kernels

For finite frequency, the sensitivity kernel has finite width (Fresnel zone). As $\omega \to \infty$:

- **Finite-frequency kernel**: $K_m(\mathbf{x})$ has width $\sim \sqrt{\lambda L}$ (Fresnel zone)
- **Ray kernel**: $K_m^T(\mathbf{x})$ collapses to a line (delta function)

The ray kernel is the $\omega \to \infty$ limit of the finite-frequency kernel.

Travel Time Sensitivity

The kernel tells us:

- Where a velocity change affects travel time: **only along the ray path**
- How much it affects travel time: proportional to $c_0(\mathbf{x})/2$
- Higher sensitivity where background velocity is higher

Connection to Other Representations

Sensitivity to Velocity c

If we want sensitivity to c rather than m:

Since $\delta m \approx -\frac{2}{c_0^3}\delta c$:

$$\begin{aligned}\delta T &\approx \int_\Omega \frac{1}{2}c_0(\mathbf{x})\left(-\frac{2}{c_0(\mathbf{x})^3}\delta c(\mathbf{x})\right)\delta(\mathbf{x}-\mathbf{x}_{\text{ray}}^{(0)})\,d\mathbf{x}\\ &= -\int_\Omega \frac{1}{c_0(\mathbf{x})^2}\delta c(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_{\text{ray}}^{(0)})\,d\mathbf{x}\end{aligned}$$

So the velocity sensitivity kernel is:

$$K_c^T(\mathbf{x}) = -\frac{1}{c_0(\mathbf{x})^2}\delta(\mathbf{x}-\mathbf{x}_{\text{ray}}^{(0)})$$

Sensitivity to Slowness $s = 1/c$

Slowness $s = 1/c = \sqrt{m}$, so $\delta s = \delta(\sqrt{m}) \approx \frac{1}{2\sqrt{m_0}}\delta m = \frac{c_0}{2}\delta m$.

Then:

$$\delta T \approx \int_\Omega \delta s(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_{\text{ray}}^{(0)})\,d\mathbf{x}$$

So:

$$K_s^T(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}_{\text{ray}}^{(0)})$$

This makes sense: travel time is simply integral of slowness along path.

Summary of Key Results

For the ray approximation ($\omega \to \infty$):

$$\boxed{\delta T \approx \frac{1}{2} \int_\Omega c_0(\mathbf{x}) \delta m(\mathbf{x})\, \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})\, d\mathbf{x}}$$

Sensitivity kernel for m:

$$\boxed{K_m^T(\mathbf{x}) = \frac{1}{2} c_0(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}_{\text{ray}}^{(0)})}$$

Important Notes

1. **High-frequency limit**: Valid when $\omega \to \infty$, wavelength $\lambda \to 0$
2. **Thin kernel**: Sensitivity only on the geometric ray path
3. **Fermat's principle**: Ray path perturbation contribution is second order

Applications

- Seismic tomography (ray-based)
- Geophysical inversion
- Medical ultrasound tomography
- Any high-frequency wave propagation problem

The ray approximation provides the simplest possible sensitivity kernel and forms the basis for many tomographic methods, though finite-frequency kernels are needed for higher resolution.