

Surface Wave (Rayleigh Wave) in Homogeneous Half Space

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1. – General Wave Equation

The 3D equations of motion for an isotropic linear-elastic medium can be written as:

$$\begin{aligned}\rho \frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x \\ \rho \frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y \\ \rho \frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z\end{aligned}\tag{1}$$

The tensor of elastic moduli for an isotropic medium is given as:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{bmatrix}$$

$$\sigma_{xx} = \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{xx}$$

$$\sigma_{yy} = \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{yy}$$

$$\sigma_{zz} = \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{zz}$$

$$\sigma_{xy} = 2\mu\epsilon_{xy}$$

$$\sigma_{yz} = 2\mu\epsilon_{yz}$$

$$\sigma_{zx} = 2\mu\epsilon_{zx}$$

$$\begin{aligned}\epsilon_{xx} &= \frac{1}{2} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x} \right) = \frac{\partial u_x}{\partial x} & \epsilon_{yz} &= \frac{1}{2} \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \\ \epsilon_{yy} &= \frac{1}{2} \left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y} \right) = \frac{\partial u_y}{\partial y} & \epsilon_{xz} &= \frac{1}{2} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \\ \epsilon_{zz} &= \frac{1}{2} \left(\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z} \right) = \frac{\partial u_z}{\partial z} & \epsilon_{xy} &= \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)\end{aligned}$$

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} + M_{ij}$$

Where:

- u_i the displacement [m]
- σ_{ij} the stress tensor [Pa]
- M_{ij} the moment tensor [Pa]
- f_i the source term [N/m³]
- ϵ_{ij} the strain tensor []
- c_{ijkl} the elastic tensor [Pa]
- ρ the density [kg/m³]

Rayleigh-type surface waves:

A Rayleigh wave is typically analyzed in a 2D plane because it propagates along a surface (the x -direction) and varies with depth (the z -direction), with no variation in the y -direction (out-of-plane). This is called **plane strain** conditions.

For **Rayleigh-type surface waves** propagating in the x - z plane (plane strain), we assume:

$$\frac{\partial}{\partial y} = 0, \quad v = u_y = 0.$$

So:

$$\begin{aligned} \rho \frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} + f_x \\ \rho \frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} + f_z \\ \sigma_{xx} &= (\lambda + 2\mu) \frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_z}{\partial z} \\ \sigma_{zz} &= \lambda \frac{\partial u_x}{\partial x} + (\lambda + 2\mu) \frac{\partial u_z}{\partial z} \\ \sigma_{xz} &= \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \end{aligned} \tag{2}$$

Rayleigh waves are surface waves that propagate along a boundary and their energy decays exponentially with depth. They propagate along the free surface of an elastic half-space (e.g., the Earth's surface). It is a key component of seismic waves generated by earthquakes.

Key Characteristics:

- **Evanescent Nature:** The wave's energy decreases exponentially with depth. It is “trapped” near the surface.
- **Elliptical Particle Motion:** Particles near the surface move in a retrograde ellipse (a vertical ellipse where the motion at the top is opposite to the direction of wave propagation). As depth increases, the ellipse becomes prograde and flattens out.
- **Non-Dispersive (in a homogeneous half-space):** Its velocity is constant and does not depend on frequency. Dispersion only occurs in layered media.
- **Slower than Body Waves:** Its velocity c is slower than both the shear wave velocity β and the compressional wave velocity α .

2. – Helmholtz Decomposition

The **Helmholtz Decomposition** states that any sufficiently smooth vector field $\mathbf{u}$ can be decomposed into a curl-free component and a divergence-free component:

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s = \nabla\phi + \nabla \times \Psi \quad (3)$$

where:

- ϕ is a **scalar potential**
- Ψ is a **vector potential**
- $\mathbf{u}_p = \nabla\phi$ is curl-free ($\nabla \times \mathbf{u}_p = 0$)
- $\mathbf{u}_s = \nabla \times \Psi$ is divergence-free ($\nabla \cdot \mathbf{u}_s = 0$)

Decoupling the Wave Equation

We now substitute the decomposition (3) into the simplified wave equation 4.

$$\boxed{\rho \ddot{\mathbf{u}} = (\lambda + 2\mu) \nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u})} \quad (4)$$

Left-Hand Side (Time Derivatives)

$$\rho \ddot{\mathbf{u}} = \rho \frac{\partial^2}{\partial t^2} (\nabla\phi + \nabla \times \Psi) = \rho \nabla \ddot{\phi} + \rho \nabla \times \ddot{\Psi} \quad (5)$$

Right-Hand Side (Spatial Derivatives)

$$(\lambda + 2\mu) \nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u})$$

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\nabla\phi + \nabla \times \Psi) = \nabla \cdot (\nabla\phi) + \nabla \cdot (\nabla \times \Psi) = \nabla^2\phi + 0 \quad (6)$$

$$\nabla \times \mathbf{u} = \nabla \times (\nabla\phi + \nabla \times \Psi) = \nabla \times (\nabla\phi) + \nabla \times (\nabla \times \Psi) = 0 + \nabla \times (\nabla \times \Psi) \quad (7)$$

Substituting (6) and (7) into the right-hand side of equation 1:

$$\begin{aligned} \text{RHS} &= (\lambda + 2\mu)\nabla(\nabla^2\phi) - \mu\nabla \times [\nabla \times (\nabla \times \Psi)] \\ &= (\lambda + 2\mu)\nabla(\nabla^2\phi) - \mu\nabla \times [\nabla(\nabla \cdot \Psi) - \nabla^2\Psi] \end{aligned} \quad (8)$$

We can choose the **Coulomb gauge** ($\nabla \cdot \Psi = 0$), which simplifies (8) to:

$$\text{RHS} = (\lambda + 2\mu)\nabla(\nabla^2\phi) + \mu\nabla \times (\nabla^2\Psi) \quad (9)$$

The Final Decoupled Form

Equating the left-hand side (5) with the right-hand side (9):

$$\nabla \left[(\lambda + 2\mu)\nabla^2\phi - \rho\ddot{\phi} \right] + \nabla \times \left[\mu\nabla^2\Psi - \rho\ddot{\Psi} \right] = 0 \quad (10)$$

For this sum of a gradient and a curl to be zero everywhere, the terms inside the brackets must each be zero (or at most equal to a constant, which can be ignored for wave solutions):

$$\nabla \left[(\lambda + 2\mu)\nabla^2\phi - \rho\ddot{\phi} \right] = 0 \quad (11)$$

$$\nabla \times \left[\mu\nabla^2\Psi - \rho\ddot{\Psi} \right] = 0 \quad (12)$$

This leads to two independent, decoupled wave equations.

Final Wave Equations and Speeds

From (11) and (12), we obtain:

P-Wave Equation (Compressional)

$$\frac{1}{\alpha^2}\ddot{\phi} = \nabla^2\phi, \quad \text{where} \quad \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad (13)$$

- α is the **P-wave speed**.
- Derived from the scalar potential ϕ .
- Particle motion is **parallel** to the direction of propagation ($\mathbf{u}_p = \nabla\phi$).

S-Wave Equation (Shear)

$$\frac{1}{\beta^2}\ddot{\Psi} = \nabla^2\Psi, \quad \text{where} \quad \beta = \sqrt{\frac{\mu}{\rho}} \quad (14)$$

- β is the **S-wave speed**.
- Derived from the vector potential Ψ .
- Particle motion is **perpendicular** to the direction of propagation ($\mathbf{u}_s = \nabla \times \Psi$).

Summary

The Helmholtz decomposition successfully decouples the elastic wave equation into two independent wave types:

- **P-waves** (faster, $\alpha > \beta$, compressional) governed by the scalar potential ϕ .
- **S-waves** (slower, shear) governed by the vector potential Ψ .

This derivation elegantly explains the fundamental separation of body waves observed in seismology.

3. – Displacement Fields

Computing Displacements at Interface from Potentials

To handle the coupled P-SV system, we use scalar potential ϕ for P-waves and vector potential ψ for SV-waves, where the displacement $\mathbf{u}$ is:

$$\mathbf{u} = \nabla\phi + \nabla \times \psi$$

P-waves (dilatational):

P-waves are **irrotational** (no rotation) and can be described by a scalar potential ϕ such that:

$$\mathbf{u}_P = \nabla\phi \tag{15}$$

In component form:

$$\boxed{u_x = \frac{\partial\phi}{\partial x}, \quad u_z = \frac{\partial\phi}{\partial z}} \tag{16}$$

No u_y component because we are considering 2D. $\frac{\partial}{\partial y} = 0$, we are dealing with a 2D problem in $x - z$ plane (plane strain)

Take Note:

Why this works

Wave equation for P-waves

$$\nabla^2 \phi = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} \quad (17)$$

Checking irrotationality

$$\nabla \times \mathbf{u}_P = \nabla \times (\nabla \phi) = 0 \quad (18)$$

This is always true because the curl of a gradient is identically zero.

Dilatation (volume change)

$$\nabla \cdot \mathbf{u}_P = \nabla \cdot (\nabla \phi) = \nabla^2 \phi \neq 0 \quad (19)$$

SV-waves (shear, vertical polarization):

SV-waves are **solenoidal** (divergence-free) and can be described by a vector potential. For 2D motion in the x-z plane, we use:

$$\mathbf{u}_{SV} = \nabla \times (0, \psi, 0) = \nabla \times (\psi \hat{y}) \quad (20)$$

For 2D P-SV case in x-z plane, the vector potential has only a y-component: $\psi = (0, \psi, 0)$.

Detailed derivation

The curl in Cartesian coordinates is:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \quad (21)$$

For $\mathbf{A} = (0, \psi, 0)$:

$$\nabla \times (0, \psi, 0) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \psi & 0 \end{vmatrix} \quad (22)$$

Computing each component:**x-component:**

$$(\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial 0}{\partial y} - \frac{\partial \psi}{\partial z} = -\frac{\partial \psi}{\partial z} \quad (23)$$

For 2D problems where nothing depends on y ($\partial/\partial y = 0$):**Standard curl calculation:**

$$(\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\frac{\partial \psi}{\partial z} \quad (24)$$

$$(\nabla \times \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0 \quad (25)$$

$$(\nabla \times \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{\partial \psi}{\partial x} \quad (26)$$

$$u_x = -\frac{\partial \psi}{\partial z}, \quad u_z = \frac{\partial \psi}{\partial x} \quad (27)$$

So:

$$\nabla \times \boldsymbol{\psi} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \psi & 0 \end{vmatrix} = -\hat{x} \frac{\partial \psi}{\partial z} + \hat{z} \frac{\partial \psi}{\partial x} \quad (28)$$

Therefore:

$$\boxed{u_x = -\frac{\partial \psi}{\partial z}, \quad u_z = +\frac{\partial \psi}{\partial x}} \quad (29)$$

Why this works

Wave equation for SV-waves

$$\nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} \quad (30)$$

Checking solenoidality (divergence-free)

$$\nabla \cdot \mathbf{u}_{SV} = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = \frac{\partial^2 \psi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial z \partial x} = 0 \quad (31)$$

SV-waves involve no volume change.

Rotation (vorticity)

$$\nabla \times \mathbf{u}_{SV} = \nabla \times (\nabla \times (0, \psi, 0)) \neq 0 \quad (32)$$

SV-waves involve rotation/shear.

So, from equation 10 and 20:

Horizontal displacement:

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \quad (33)$$

Vertical displacement:

$$u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \quad (34)$$

In matrix form, the continuity of displacement at the interface becomes:

$$\begin{pmatrix} u_x \\ u_z \end{pmatrix} = \begin{pmatrix} \partial_x & -\partial_z \\ \partial_z & \partial_x \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \quad (35)$$

4. – Proposed Wave Solutions

We are looking for a wave propagating in the x -direction with velocity c , and decaying in the z -direction. This means the amplitude of the wave decays with depth. The general solutions for the potentials are:

$$\phi(x, z, t) = Ae^{i(\omega t - k_x x - k_{z\alpha} z)}$$

$$\psi(x, z, t) = Be^{i(\omega t - k_x x - k_{z\beta} z)}$$

Where:

- ω is the angular frequency.
- k_x is the horizontal wavenumber ($k_x = \omega/c$).
- $k_{z\alpha}$ and $k_{z\beta}$ are the vertical wavenumbers for the P and S components, respectively.
- A and B are amplitudes to be determined.

The Evanescent (Decaying) Condition

For the wave to be a surface wave, it must decay with depth ($z \rightarrow +\infty$). This means the solution for $F(z)$ must be exponential decay, not oscillation. This requires the vertical wavenumbers to be **imaginary**. We define them as:

$$k_{z\alpha} = -ik_x r_\alpha, \quad k_{z\beta} = -ik_x r_\beta$$

where r_α and r_β are real, positive numbers that ensure exponential decay: $e^{-k_x r_\alpha z}$.

Such that:

$$\phi(x, z, t) = Ae^{i(-k_{z\alpha} z)} e^{i(\omega t - k_x x)} = Ae^{-k_x r_\alpha z} e^{i(\omega t - k_x x)}$$

$$\psi(x, z, t) = Be^{i(-k_{z\beta} z)} e^{i(\omega t - k_x x)} = Be^{-k_x r_\beta z} e^{i(\omega t - k_x x)}$$

Substitute each ansatz into its wave equation 17 and 30 to get the dispersion relations for the vertical wavenumbers.

$$\nabla^2 \phi = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} \tag{36}$$

$$\nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} \tag{37}$$

Given:

$$\phi(x, z, t) = Ae^{-k_x r_\alpha z} e^{i(\omega t - k_x x)}, \quad \psi(x, z, t) = Be^{-k_x r_\beta z} e^{i(\omega t - k_x x)}$$

Substitute into

$$\nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2}, \quad \nabla^2 \phi = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2}.$$

Assuming no y -dependence, we have $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$.

For $\psi(x, z, t)$:

$$\psi(x, z, t) = Be^{-k_x r_\beta z} e^{i(\omega t - k_x x)}.$$

Compute derivatives:

$$\frac{\partial^2 \psi}{\partial x^2} = (-k_x^2) \psi, \quad \frac{\partial^2 \psi}{\partial z^2} = (k_x^2 r_\beta^2) \psi.$$

Thus,

$$\boxed{\nabla^2 \psi = k_x^2 (r_\beta^2 - 1) \psi}$$

Time derivative:

$$\frac{\partial^2 \psi}{\partial t^2} = (i\omega)^2 \psi = -\omega^2 \psi.$$

Substitute into the wave equation:

$$k_x^2 (r_\beta^2 - 1) \psi = -\frac{\omega^2}{\beta^2} \psi.$$

Cancel ψ and rearrange:

$$\boxed{r_\beta^2 = 1 - \frac{\omega^2}{\beta^2 k_x^2}}$$

For $\phi(x, z, t)$:

$$\phi(x, z, t) = Ae^{-k_x r_\alpha z} e^{i(\omega t - k_x x)}.$$

Similarly,

$$\nabla^2 \phi = k_x^2 (r_\alpha^2 - 1) \phi, \quad \frac{\partial^2 \phi}{\partial t^2} = -\omega^2 \phi.$$

Substitute into the corresponding equation:

$$k_x^2 (r_\alpha^2 - 1) \phi = -\frac{\omega^2}{\alpha^2} \phi.$$

Simplifying:

$$\boxed{r_\alpha^2 = 1 - \frac{\omega^2}{\alpha^2 k_x^2}}$$

Therefore,

$$r_\alpha^2 = 1 - \frac{\omega^2}{\alpha^2 k_x^2}, \quad r_\beta^2 = 1 - \frac{\omega^2}{\beta^2 k_x^2}$$

Since: $k_x = \omega/c$, $k_x^2 = \omega^2/c^2$

$$\boxed{r_\alpha^2 = 1 - \frac{c^2}{\alpha^2}, \quad r_\beta^2 = 1 - \frac{c^2}{\beta^2}}$$

Remember:

$$k_{z\alpha} = -ik_x r_\alpha, \quad k_{z\beta} = -ik_x r_\beta$$

$$r_\alpha = \frac{ik_{z\alpha}}{k_x}, \quad r_\beta = \frac{ik_{z\beta}}{k_x}$$

$$\boxed{k_{z\alpha} = -ik_x \sqrt{1 - \frac{c^2}{\alpha^2}}, \quad k_{z\beta} = -ik_x \sqrt{1 - \frac{c^2}{\beta^2}}}$$

Compute Displacement u_x , u_z

Given potentials:

$$\phi(x, z, t) = A e^{-k_x r_\alpha z} e^{i(\omega t - k_x x)}, \quad \psi(x, z, t) = B e^{-k_x r_\beta z} e^{i(\omega t - k_x x)}.$$

Horizontal displacement:

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}.$$

Compute each term:

$$\frac{\partial \phi}{\partial x} = -ik_x A e^{-k_x r_\alpha z} e^{i(\omega t - k_x x)}, \quad \frac{\partial \psi}{\partial z} = -k_x r_\beta B e^{-k_x r_\beta z} e^{i(\omega t - k_x x)}.$$

Therefore,

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} = (-ik_x A e^{-k_x r_\alpha z} + k_x r_\beta B e^{-k_x r_\beta z}) e^{i(\omega t - k_x x)}.$$

Simplify:

$$u_x = k_x (-iA e^{-k_x r_\alpha z} + r_\beta B e^{-k_x r_\beta z}) e^{i(\omega t - k_x x)}.$$

Vertical displacement:

$$u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}.$$

Compute each term:

$$\frac{\partial \phi}{\partial z} = -k_x r_\alpha A e^{-k_x r_\alpha z} e^{i(\omega t - k_x x)}, \quad \frac{\partial \psi}{\partial x} = -ik_x B e^{-k_x r_\beta z} e^{i(\omega t - k_x x)}$$

Hence,

$$u_z = (-k_x r_\alpha A e^{-k_x r_\alpha z} - ik_x B e^{-k_x r_\beta z}) e^{i(\omega t - k_x x)}$$

Simplify:

$$u_z = -k_x (r_\alpha A e^{-k_x r_\alpha z} + iB e^{-k_x r_\beta z}) e^{i(\omega t - k_x x)}$$

Final Results:

$$\begin{aligned} u_x &= k_x (-iA e^{-k_x r_\alpha z} + r_\beta B e^{-k_x r_\beta z}) e^{i(\omega t - k_x x)} \\ u_z &= -k_x (r_\alpha A e^{-k_x r_\alpha z} + iB e^{-k_x r_\beta z}) e^{i(\omega t - k_x x)} \end{aligned}$$

Stress components and free-surface boundary conditions

The free surface at $z = 0$ has zero stress:

$$\sigma_{zz}(x, 0, t) = 0 \quad \text{and} \quad \sigma_{zx}(x, 0, t) = 0$$

(Note: σ_{xx} is sometimes used, but for an isotropic medium, the vanishing of σ_{zx} and σ_{zz} is the standard and sufficient condition).

We need the stress-strain and strain-displacement relations:

- $\sigma_{zz} = \lambda(\epsilon_{xx} + \epsilon_{zz}) + 2\mu\epsilon_{zz}$
- $\sigma_{zx} = 2\mu\epsilon_{zx}$
- $\epsilon_{xx} = \partial_x u_x, \epsilon_{zz} = \partial_z u_z, \epsilon_{zx} = \frac{1}{2}(\partial_z u_x + \partial_x u_z)$

$$\begin{aligned}\sigma_{xx} &= (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z} \\ \sigma_{zz} &= \lambda\frac{\partial u_x}{\partial x} + (\lambda + 2\mu)\frac{\partial u_z}{\partial z} \\ \sigma_{xz} &= \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\end{aligned}\tag{38}$$

We begin with the definitions and solutions:

Displacements from Potentials

$$u_x = \partial_x \phi - \partial_z \psi$$

$$u_z = \partial_z \phi + \partial_x \psi$$

Proposed Solutions

$$\phi = Ae^{i(\omega t - k_x x - k_{z\alpha} z)} = Ae^{i(\omega t - k_x x)}e^{-ik_{z\alpha} z}$$

$$\psi = Be^{i(\omega t - k_x x - k_{z\beta} z)} = Be^{i(\omega t - k_x x)}e^{-ik_{z\beta} z}$$

For surface waves, the vertical wavenumbers are imaginary to ensure decay:

$$k_{z\alpha} = -ik_x r_\alpha, \quad k_{z\beta} = -ik_x r_\beta$$

with

$$r_\alpha = \sqrt{1 - c^2/\alpha^2}, \quad r_\beta = \sqrt{1 - c^2/\beta^2}$$

Thus, the potentials become:

$$\phi = Ae^{i(\omega t - k_x x)}e^{-k_x r_\alpha z}$$

$$\psi = Be^{i(\omega t - k_x x)}e^{-k_x r_\beta z}$$

The term $e^{i(\omega t - k_x x)}$ is common to all subsequent expressions and will be omitted for clarity during the differentiation process.

Compute the Displacement Components u_x and u_z

Let's compute the derivatives of the potentials.

For $\phi = Ae^{-k_x r_\alpha z} e^{-ik_x x}$:

$$\begin{aligned}\partial_x \phi &= -ik_x \phi \\ \partial_z \phi &= -k_x r_\alpha \phi\end{aligned}$$

For $\psi = Be^{-k_x r_\beta z} e^{-ik_x x}$:

$$\begin{aligned}\partial_x \psi &= -ik_x \psi \\ \partial_z \psi &= -k_x r_\beta \psi\end{aligned}$$

Now substitute into the displacement equations:

u_x Component:

$$\begin{aligned}u_x &= \partial_x \phi - \partial_z \psi = (-ik_x \phi) - (-k_x r_\beta \psi) \\ &= -ik_x \phi + k_x r_\beta \psi \\ \Rightarrow u_x &= k_x (-iAe^{-k_x r_\alpha z} + r_\beta Be^{-k_x r_\beta z}) e^{-ik_x x}\end{aligned}$$

u_z Component:

$$\begin{aligned}u_z &= \partial_z \phi + \partial_x \psi = (-k_x r_\alpha \phi) + (-ik_x \psi) \\ &= -k_x r_\alpha \phi - ik_x \psi \\ \Rightarrow u_z &= k_x (-r_\alpha Ae^{-k_x r_\alpha z} - iBe^{-k_x r_\beta z}) e^{-ik_x x}\end{aligned}$$

Compute the Strain Components ϵ_{xx} , ϵ_{zz} , ϵ_{zx}

We now differentiate the displacements. Note that for our solutions:

$$\partial_x \equiv -ik_x$$

Strain $\epsilon_{xx} = \partial_x u_x$:

$$\begin{aligned}
 \epsilon_{xx} &= \partial_x u_x = -ik_x u_x \\
 &= -ik_x [k_x(-iAe^{-k_x r_\alpha z} + r_\beta B e^{-k_x r_\beta z})] \\
 &= k_x^2 [(-i)(-i)Ae^{-k_x r_\alpha z} + (-i)r_\beta B e^{-k_x r_\beta z}] \\
 &= k_x^2 [(-1)Ae^{-k_x r_\alpha z} - ir_\beta B e^{-k_x r_\beta z}]
 \end{aligned}$$

$$\boxed{\epsilon_{xx} = -k_x^2 A e^{-k_x r_\alpha z} - ik_x^2 r_\beta B e^{-k_x r_\beta z}}$$

Strain $\epsilon_{zz} = \partial_z u_z$:

$$\begin{aligned}
 u_z &= k_x(-r_\alpha A e^{-k_x r_\alpha z} - iB e^{-k_x r_\beta z})e^{-ik_x x} \\
 \partial_z u_z &= k_x [-r_\alpha A(-k_x r_\alpha)e^{-k_x r_\alpha z} - iB(-k_x r_\beta)e^{-k_x r_\beta z}] e^{-ik_x x} \\
 &= k_x^2 [r_\alpha^2 A e^{-k_x r_\alpha z} + ir_\beta B e^{-k_x r_\beta z}] e^{-ik_x x}
 \end{aligned}$$

$$\boxed{\epsilon_{zz} = k_x^2 r_\alpha^2 A e^{-k_x r_\alpha z} + ik_x^2 r_\beta B e^{-k_x r_\beta z}}$$

Recall $r_\alpha^2 = 1 - c^2/\alpha^2$.

Strain $\epsilon_{zx} = \frac{1}{2}(\partial_z u_x + \partial_x u_z)$:

First, $\partial_z u_x$:

$$\begin{aligned}
 u_x &= k_x(-iAe^{-k_x r_\alpha z} + r_\beta B e^{-k_x r_\beta z})e^{-ik_x x} \\
 \partial_z u_x &= k_x [-iA(-k_x r_\alpha)e^{-k_x r_\alpha z} + r_\beta B(-k_x r_\beta)e^{-k_x r_\beta z}] e^{-ik_x x} \\
 &= k_x^2 [ir_\alpha A e^{-k_x r_\alpha z} - r_\beta^2 B e^{-k_x r_\beta z}] e^{-ik_x x}
 \end{aligned}$$

Second, $\partial_x u_z$:

$$\begin{aligned}
 u_z &= k_x(-r_\alpha A e^{-k_x r_\alpha z} - iB e^{-k_x r_\beta z})e^{-ik_x x} \\
 \partial_x u_z &= -ik_x u_z = -ik_x [k_x(-r_\alpha A e^{-k_x r_\alpha z} - iB e^{-k_x r_\beta z})] \\
 &= k_x^2 [ir_\alpha A e^{-k_x r_\alpha z} - B e^{-k_x r_\beta z}] e^{-ik_x x}
 \end{aligned}$$

Now add them for ϵ_{zx} :

$$\begin{aligned}\partial_z u_x + \partial_x u_z &= k_x^2 [ir_\alpha A + ir_\alpha A] e^{-k_x r_\alpha z} + k_x^2 [-r_\beta^2 B - B] e^{-k_x r_\beta z} \\ &= k_x^2 [2ir_\alpha A e^{-k_x r_\alpha z} - (r_\beta^2 + 1) B e^{-k_x r_\beta z}]\end{aligned}$$

Therefore,

$$\begin{aligned}\epsilon_{zx} &= \frac{1}{2}(\partial_z u_x + \partial_x u_z) \\ &= \frac{k_x^2}{2} [2ir_\alpha A e^{-k_x r_\alpha z} - (r_\beta^2 + 1) B e^{-k_x r_\beta z}]\end{aligned}$$

$$\boxed{\epsilon_{zx} = ik_x^2 r_\alpha A e^{-k_x r_\alpha z} - \frac{k_x^2}{2} (r_\beta^2 + 1) B e^{-k_x r_\beta z}}$$

Recall $r_\beta^2 = 1 - c^2/\beta^2$, so $r_\beta^2 + 1 = 2 - c^2/\beta^2$.

Compute the Stress Components σ_{zz} and σ_{zx}

Stress $\sigma_{zz} = \lambda(\epsilon_{xx} + \epsilon_{zz}) + 2\mu\epsilon_{zz} = \lambda\epsilon_{xx} + (\lambda + 2\mu)\epsilon_{zz}$

First, find $\epsilon_{xx} + \epsilon_{zz}$:

$$\begin{aligned}\epsilon_{xx} + \epsilon_{zz} &= [-k_x^2 A e^{-k_x r_\alpha z} - ik_x^2 r_\beta B e^{-k_x r_\beta z}] \\ &\quad + [k_x^2 r_\alpha^2 A e^{-k_x r_\alpha z} + ik_x^2 r_\beta B e^{-k_x r_\beta z}] \\ &= k_x^2 A (r_\alpha^2 - 1) e^{-k_x r_\alpha z}\end{aligned}$$

But $r_\alpha^2 - 1 = (1 - c^2/\alpha^2) - 1 = -c^2/\alpha^2$, so:

$$\epsilon_{xx} + \epsilon_{zz} = -k_x^2 A (c^2/\alpha^2) e^{-k_x r_\alpha z}$$

Now compute σ_{zz} :

$$\begin{aligned}\sigma_{zz} &= \lambda(\epsilon_{xx} + \epsilon_{zz}) + 2\mu\epsilon_{zz} \\ &= \lambda [-k_x^2 A (c^2/\alpha^2) e^{-k_x r_\alpha z}] + 2\mu [k_x^2 r_\alpha^2 A e^{-k_x r_\alpha z} + ik_x^2 r_\beta B e^{-k_x r_\beta z}] \\ &= k_x^2 A [-\lambda(c^2/\alpha^2) + 2\mu r_\alpha^2] e^{-k_x r_\alpha z} + 2\mu [ik_x^2 r_\beta B e^{-k_x r_\beta z}]\end{aligned}$$

Simplify the coefficient of A using:

$$\lambda = \rho\alpha^2 - 2\mu$$

$$r_\alpha^2 = 1 - c^2/\alpha^2$$

Then:

$$\begin{aligned} & -\lambda(c^2/\alpha^2) + 2\mu r_\alpha^2 \\ &= -(\rho\alpha^2 - 2\mu)(c^2/\alpha^2) + 2\mu(1 - c^2/\alpha^2) \\ &= -\rho c^2 + 2\mu(c^2/\alpha^2) + 2\mu - 2\mu(c^2/\alpha^2) \\ &= -\rho c^2 + 2\mu \end{aligned}$$

Using $\rho = \mu/\beta^2$, we get:

$$-\rho c^2 + 2\mu = -\frac{\mu}{\beta^2}c^2 + 2\mu = \mu \left(2 - \frac{c^2}{\beta^2} \right)$$

Therefore:

$$\sigma_{zz} = \mu k_x^2 \left[\left(2 - \frac{c^2}{\beta^2} \right) A e^{-k_x r_\alpha z} + 2i r_\beta B e^{-k_x r_\beta z} \right]$$

Stress $\sigma_{zx} = 2\mu\epsilon_{zx}$:

$$\begin{aligned} \sigma_{zx} &= 2\mu \left[i k_x^2 r_\alpha A e^{-k_x r_\alpha z} - \frac{k_x^2}{2} (r_\beta^2 + 1) B e^{-k_x r_\beta z} \right] \\ &= 2\mu k_x^2 \left[i r_\alpha A e^{-k_x r_\alpha z} - \frac{1}{2} (2 - c^2/\beta^2) B e^{-k_x r_\beta z} \right] \end{aligned}$$

$$\boxed{\sigma_{zx} = 2\mu k_x^2 \left[i r_\alpha A e^{-k_x r_\alpha z} - \frac{1}{2} \left(2 - \frac{c^2}{\beta^2} \right) B e^{-k_x r_\beta z} \right]}$$

Apply Boundary Conditions at $z = 0$

At $z = 0$, $e^{-k_x r_\alpha z} = e^{-k_x r_\beta z} = 1$.

BC 1: $\sigma_{zz}(z = 0) = 0$

$$\begin{aligned}\mu k_x^2 \left[\left(2 - \frac{c^2}{\beta^2} \right) A + 2ir_\beta B \right] &= 0 \\ \left(2 - \frac{c^2}{\beta^2} \right) A + 2ir_\beta B &= 0\end{aligned}\tag{39}$$

BC 2: $\sigma_{zx}(z=0) = 0$

$$\begin{aligned}2\mu k_x^2 \left[ir_\alpha A - \frac{1}{2} \left(2 - \frac{c^2}{\beta^2} \right) B \right] &= 0 \\ ir_\alpha A - \frac{1}{2} \left(2 - \frac{c^2}{\beta^2} \right) B &= 0 \\ 2ir_\alpha A - \left(2 - \frac{c^2}{\beta^2} \right) B &= 0\end{aligned}\tag{40}$$

Write the Homogeneous System

We have Eq. 39 and Eq. 40:

$$\begin{cases} \left(2 - \frac{c^2}{\beta^2} \right) A + 2i\sqrt{1 - \frac{c^2}{\beta^2}} B = 0 \\ 2i\sqrt{1 - \frac{c^2}{\alpha^2}} A - \left(2 - \frac{c^2}{\beta^2} \right) B = 0 \end{cases}$$

In matrix form:

$$\begin{pmatrix} \left(2 - \frac{c^2}{\beta^2} \right) & 2i\sqrt{1 - \frac{c^2}{\beta^2}} \\ 2i\sqrt{1 - \frac{c^2}{\alpha^2}} & -\left(2 - \frac{c^2}{\beta^2} \right) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is equivalent to the system in the notes:

$$\begin{pmatrix} 2\sqrt{1 - c^2/\alpha^2} & -(2 - c^2/\beta^2) \\ (2 - c^2/\beta^2) & 2\sqrt{1 - c^2/\beta^2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is obtained by multiplying the first row by $-i$ and the second row by i , then swapping rows. This is a mathematically equivalent transformation that makes the matrix look symmetric and avoids explicit imaginary units, as the amplitudes A and B can absorb phase factors.

Note: The signs in the matrix can vary slightly depending on the sign convention used for the vertical wavenumbers, but the final characteristic equation is always the same

The Characteristic (Rayleigh) Equation

For a non-trivial solution $(A, B) \neq (0, 0)$, the determinant of the coefficient matrix must be zero:

$$\left(2\sqrt{1 - \frac{c^2}{\alpha^2}}\right) \left(2\sqrt{1 - \frac{c^2}{\beta^2}}\right) + \left(2 - \frac{c^2}{\beta^2}\right)^2 = 0$$

This is the **Rayleigh Equation**. It is more commonly written by squaring both sides to eliminate the square roots (which introduces extraneous roots that must be checked):

$$\left(2 - \frac{c^2}{\beta^2}\right)^2 = 4\sqrt{1 - \frac{c^2}{\alpha^2}}\sqrt{1 - \frac{c^2}{\beta^2}}$$

Squaring both sides gives the polynomial form:

$$\left(2 - \frac{c^2}{\beta^2}\right)^4 = 16 \left(1 - \frac{c^2}{\alpha^2}\right) \left(1 - \frac{c^2}{\beta^2}\right) \quad (41)$$

Solving for the Rayleigh Wave Velocity c

This equation has multiple solutions for c , but we are only interested in the physical one where $c < \beta < \alpha$ (so the wave is slower than body waves and the vertical wavenumbers are imaginary, ensuring decay).

For a **Poisson solid**, a common approximation in geophysics where $\lambda = \mu$ (which implies $\nu = 0.25$ and $\alpha^2/\beta^2 = 3$), the equation simplifies. The solution is:

$$c \approx 0.9194\beta$$

This velocity is **independent of frequency**, meaning the homogeneous half-space model is **non-dispersive**.

When is it dispersive? Rayleigh waves become dispersive when propagating in a **layered medium**, such as the Earth's crust over a mantle. Different frequencies then sample different depths, and thus “feel” different average material properties, leading to a frequency-dependent velocity $c(\omega)$.

Solve for a Poisson Solid from Equation 41

A Poisson solid is defined by $\lambda = \mu$, which leads to a specific relationship between P-wave and S-wave velocities:

$$\alpha^2 = \frac{\lambda + 2\mu}{\rho} = \frac{\mu + 2\mu}{\rho} = \frac{3\mu}{\rho}$$

$$\beta^2 = \frac{\mu}{\rho}$$

$$\Rightarrow \frac{\alpha^2}{\beta^2} = 3 \quad \text{or} \quad \alpha^2 = 3\beta^2$$

Let's substitute $\gamma = \frac{c^2}{\beta^2}$ and $\frac{c^2}{\alpha^2} = \frac{c^2}{3\beta^2} = \frac{\gamma}{3}$ into the squared characteristic equation (Eq. 41):

$$(2 - \gamma)^4 = 16 \left(1 - \frac{\gamma}{3}\right) (1 - \gamma)$$

This is an equation in one variable, γ . Let's simplify and solve it.

$$(2 - \gamma)^4 = 16 \left(1 - \frac{\gamma}{3}\right) (1 - \gamma)$$

$$(2 - \gamma)^4 = 16 \left(\frac{3 - \gamma}{3}\right) (1 - \gamma)$$

$$(2 - \gamma)^4 = \frac{16}{3}(3 - \gamma)(1 - \gamma)$$

Multiply both sides by 3:

$$3(2 - \gamma)^4 = 16(3 - \gamma)(1 - \gamma)$$

Now we expand both sides. First, expand $(3 - \gamma)(1 - \gamma)$:

$$(3 - \gamma)(1 - \gamma) = 3 - 3\gamma - \gamma + \gamma^2 = 3 - 4\gamma + \gamma^2$$

So the equation becomes:

$$3(2 - \gamma)^4 = 16(3 - 4\gamma + \gamma^2) \tag{42}$$

Now, let $p = 2 - \gamma$, so $\gamma = 2 - p$. Substituting is messy, so let's expand $(2 - \gamma)^4$:

$$(2 - \gamma)^4 = \gamma^4 - 8\gamma^3 + 24\gamma^2 - 32\gamma + 16$$

Substitute back into Eq. 42:

$$3(\gamma^4 - 8\gamma^3 + 24\gamma^2 - 32\gamma + 16) = 16(\gamma^2 - 4\gamma + 3)$$

$$3\gamma^4 - 24\gamma^3 + 72\gamma^2 - 96\gamma + 48 = 16\gamma^2 - 64\gamma + 48$$

Bring all terms to one side:

$$3\gamma^4 - 24\gamma^3 + 72\gamma^2 - 96\gamma + 48 - 16\gamma^2 + 64\gamma - 48 = 0$$

$$3\gamma^4 - 24\gamma^3 + (72 - 16)\gamma^2 + (-96 + 64)\gamma + (48 - 48) = 0$$

$$3\gamma^4 - 24\gamma^3 + 56\gamma^2 - 32\gamma = 0$$

Factor out γ :

$$\gamma(3\gamma^3 - 24\gamma^2 + 56\gamma - 32) = 0$$

One solution is $\gamma = 0$, which corresponds to $c = 0$ (no wave). We are interested in the roots of the cubic:

$$3\gamma^3 - 24\gamma^2 + 56\gamma - 32 = 0$$

We can try simple rational roots. Let's try $\gamma = 2$:

$$3(8) - 24(4) + 56(2) - 32 = 24 - 96 + 112 - 32 = 8 \quad (\text{Not zero})$$

Let's try $\gamma = 4$:

$$3(64) - 24(16) + 56(4) - 32 = 192 - 384 + 224 - 32 = 0 \quad (\text{Yes!})$$

So, $\gamma = 4$ is a root. Let's perform polynomial division. Divide the cubic by $(\gamma - 4)$.

The quotient is:

$$3\gamma^2 - 12\gamma + 8$$

So the factored cubic is:

$$(\gamma - 4)(3\gamma^2 - 12\gamma + 8) = 0$$

Now solve the quadratic $3\gamma^2 - 12\gamma + 8 = 0$:

$$\gamma = \frac{12 \pm \sqrt{144 - 96}}{6} = \frac{12 \pm \sqrt{48}}{6} = \frac{12 \pm 4\sqrt{3}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$$

$$\gamma \approx 2 \pm 1.155 = 3.155, \quad 0.845$$

So our four potential solutions for $\gamma = c^2/\beta^2$ are:

1. $\gamma_1 = 0 \Rightarrow c = 0$
2. $\gamma_2 = 4 \Rightarrow c = 2\beta$
3. $\gamma_3 \approx 3.155 \Rightarrow c \approx 1.776\beta$
4. $\gamma_4 \approx 0.845 \Rightarrow c \approx 0.919\beta$

Select the Physically Valid Solution

Recall the decay condition: for the wave to be a surface wave (evanescent), we require $r_\alpha^2 > 0$ and $r_\beta^2 > 0$, meaning:

$$r_\alpha^2 = 1 - \frac{c^2}{\alpha^2} > 0 \quad \text{and} \quad r_\beta^2 = 1 - \frac{c^2}{\beta^2} > 0$$

Since $\alpha > \beta$, the stricter condition is $c < \beta$. This implies $\gamma = c^2/\beta^2 < 1$.

Let's check our solutions:

- $\gamma_1 = 0$: $c = 0$. Trivial, no wave.
- $\gamma_2 = 4$: $c = 2\beta$. Violates $c < \beta$.
- $\gamma_3 \approx 3.155$: $c \approx 1.776\beta$. Violates $c < \beta$.
- $\gamma_4 \approx 0.845$: $c \approx 0.919\beta$. **Satisfies** $c < \beta$.

Therefore, the only physically valid solution for a Poisson solid is:

$$\frac{c^2}{\beta^2} \approx 0.845 \quad \Rightarrow \quad c \approx \sqrt{0.845}\beta \approx 0.919\beta$$

This matches the value given in the notes.

Final Summary

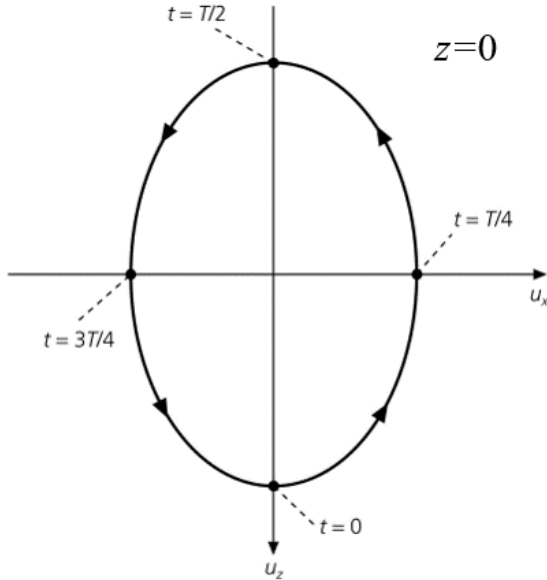
The homogeneous system has a non-trivial solution only when the Rayleigh wave velocity c satisfies the characteristic equation. For a Poisson solid ($\alpha/\beta = \sqrt{3}$), the valid solution is:

$$c \approx 0.919\beta$$

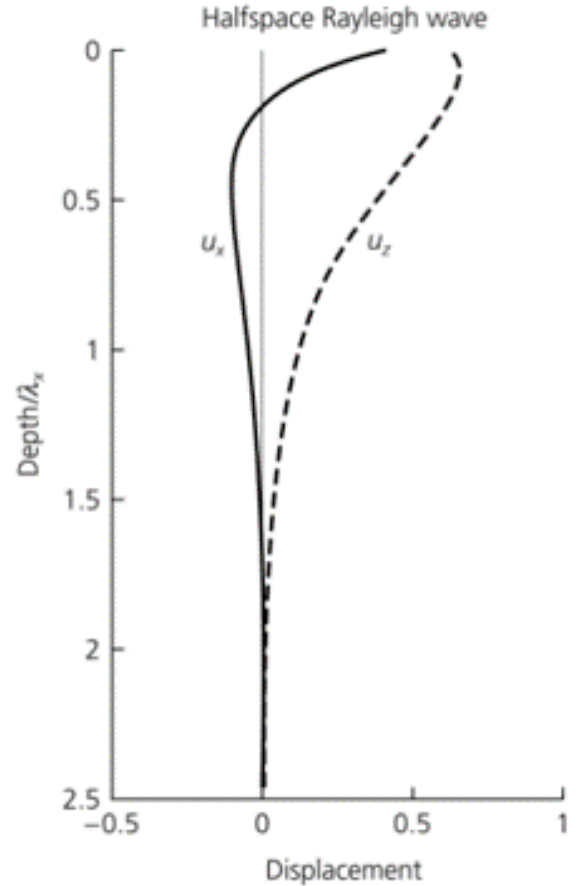
This confirms the existence of a Rayleigh wave that propagates more slowly than the shear wave and decays exponentially with depth.

Rayleigh wave motion (Poisson solid)

$$\begin{pmatrix} u_x \\ u_z \end{pmatrix} = Ak_x \begin{pmatrix} \sin(\omega t - k_x x) \\ \cos(\omega t - k_x x) \end{pmatrix} \begin{pmatrix} 1 & -0.58 \\ -0.85 & 1.47 \end{pmatrix} \begin{pmatrix} e^{-0.85k_x z} \\ e^{-0.39k_x z} \end{pmatrix}$$



(a) Rayleigh Motion



(b) Halfspace Rayleigh Wave

Figure 1: Models used in the homogeneous case

Explanation of the Wave Motion Diagram

The final part of the notes describes the particle motion.

- **Depth Decay:** The terms $e^{-0.85k_x z}$ and $e^{-0.39k_x z}$ show that both horizontal (u_x) and vertical (u_z) displacements decay exponentially with depth. The S-wave component (associated with ψ and the 0.39 term) decays more slowly than the P-wave component (associated with ϕ and the 0.85 term).
- **Elliptical Polarization:** The matrix and the phase relationship between the sine and cosine terms indicate that the motion is elliptical. The particle path over one cycle traces an ellipse.
- **Retrograde Motion:** At the surface ($z = 0$), the ellipse is **retrograde**. This means that as the wave passes, a particle moves in a vertical ellipse in the direction *opposite* to the wave propagation.
- **Phase Shifts with Depth:** The diagrams at different times ($t = T/2, T/4$, etc.) show how the particle at a fixed depth moves through its elliptical path over time. The relative amplitudes and phases of u_x and u_z change with depth, which can cause the ellipse to reverse its sense of rotation (become prograde) at a certain depth.

In summary, the notes provide a compact derivation of the fundamental properties of Rayleigh waves: their existence condition (the Rayleigh equation), their non-dispersive nature in a simple model, and their characteristic elliptical, decaying motion.