P-SV Wave Reflection and Transmission (R/T) Coefficients

October, 2025

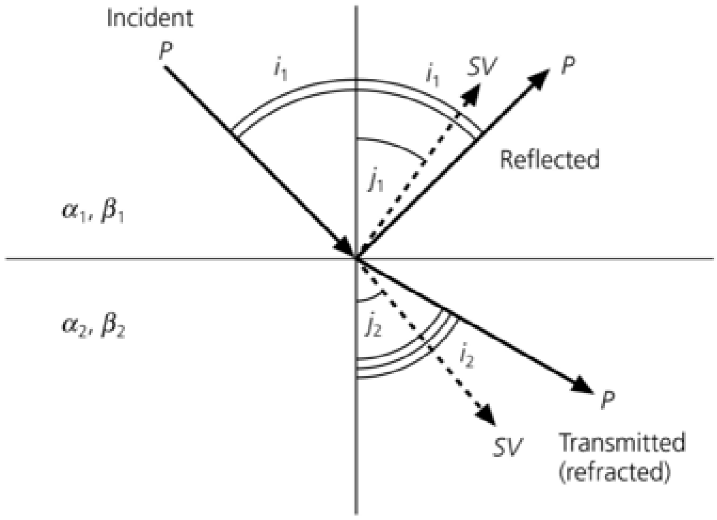

Figure 1: P-SV R/T Coefficients

This note covers **P-SV wave reflection and transmission (R/T) coefficients**, from the **full Zoeppritz equations** to their **linearized form (Shuey's equation)** and the special case of the free surface. The P-SV problem is significantly more complex than the SH-case because P-waves and SV-waves couple at an interface. When a P-wave is incident on an interface between two elastic media, it generates four waves (1):

- **Reflected P-wave** (travels back in medium 1)
- **Reflected SV-wave** (mode conversion in medium 1)
- **Transmitted P-wave** (refracted into medium 2)
- **Transmitted SV-wave** (mode conversion in medium 2)

This is fundamentally different from SH waves, where no mode conversion occurs because SH motion is perpendicular to the plane of incidence. This leads to a system of four equations instead of two.

1. Concept Overview of P-Sv Wave Propagation

Problem Setup

We consider an interface between two elastic solids:

- **Medium 1:** P-wave velocity α_1, S-wave velocity β_1, density ρ_1, Lamé parameters λ_1, μ_1
- **Medium 2:** P-wave velocity α_2, S-wave velocity β_2, density ρ_2, Lamé parameters λ_2, μ_2

Snell's Law

For an incident plane wave (P or SV), Snell's Law governs of all generated waves:

$$\frac{\sin i}{\alpha_1} = \frac{\sin i'}{\alpha_1} = \frac{\sin j}{\beta_1} = \frac{\sin i_2}{\alpha_2} = \frac{\sin j_2}{\beta_2} = p$$

where p is the ray parameter, constant for all waves.

Potential Functions

To handle the coupled P-SV system, we use scalar potential ϕ for P-waves and vector potential ψ for SV-waves, where the displacement $\mathbf{u}$ is:

$$\mathbf{u} = \nabla\phi + \nabla \times \psi$$

P-waves (dilatational):

$$\mathbf{u}_P = \nabla\phi \tag{1}$$

SV-waves (shear, vertical polarization):

$$\mathbf{u}_{SV} = \nabla \times (0, \psi, 0) \tag{2}$$

For 2D P-SV case in x-z plane, the vector potential has only a y-component: $\psi = (0, \psi, 0)$.

Boundary Conditions

At interface $z = 0$, we require continuity of:

1. **Displacement:** u_x and u_z

$$\begin{pmatrix} u_x \\ u_z \end{pmatrix} = \begin{pmatrix} \partial_x & -\partial_z \\ \partial_z & \partial_x \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{3}$$

This leads to the displacement components:

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, \quad u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}$$

2. **Stress:** σ_{zz} (normal stress) and σ_{xz} (shear stress)

Stress expressions in terms of potentials:

Remember:

$$\sigma_{ij} = \lambda \, \delta_{ij} \, \epsilon_{kk} \, + 2\mu\epsilon_{ij}$$

$$\epsilon_{kk} = \nabla \cdot u = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$$

$$\sigma_{zz} = \lambda \nabla^2 \phi + 2\mu \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right)$$

$$\sigma_{xz} = \mu \left(2 \frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} \right)$$

The Zoeppritz Equations:

Applying these four boundary conditions to the complete wavefield (incident P, reflected P, reflected SV, transmitted P, transmitted SV) results in a 4x4 system of linear equations. This full system is the Zoeppritz equations. Solving them gives the exact R/T coefficients for a planar interface between two solids. They are the "ground truth" but are algebraically complex and non-intuitive.

2. Wave Equations in Each Medium

Medium 1:

$$\phi_1 = A_1 e^{i(\omega t - k_x x + k_{z\alpha} z)} + A' e^{i(\omega t - k_x x - k_{z\alpha} z)} \tag{4}$$

$$\psi_1 = B_1 e^{i(\omega t - k_x x - k_{z\beta} z)} \tag{5}$$

Medium 2:

$$\phi_2 = A_2 e^{i(\omega t - k_x x + k_{z\alpha} z)} \tag{6}$$

$$\psi_2 = B_2 e^{i(\omega t - k_x x + k_{z\beta} z)} \tag{7}$$

Sign conventions:

- Positive k_z indicates downward propagation
- Negative k_z indicates upward propagation

Wavenumber Relations

From Snell's law, k_x is continuous across the interface:

$$k_{z\alpha} = \sqrt{\frac{\omega^2}{\alpha^2} - k_x^2}, \quad k_{z\beta} = \sqrt{\frac{\omega^2}{\beta^2} - k_x^2} \tag{8}$$

3. Displacement Fields

Computing Displacements at Interface from Potentials

To handle the coupled P-SV system, we use scalar potential ϕ for P-waves and vector potential ψ for SV-waves, where the displacement $\mathbf{u}$ is:

$$\mathbf{u} = \nabla\phi + \nabla \times \psi$$

P-waves (dilatational):

P-waves are **irrotational** (no rotation) and can be described by a scalar potential ϕ such that:

$$\mathbf{u}_P = \nabla\phi \tag{9}$$

In component form:

$$\boxed{u_x = \frac{\partial\phi}{\partial x}, \quad u_z = \frac{\partial\phi}{\partial z}} \tag{10}$$

No u_y component because we are considering 2D.

Take Note:

Why this works

Wave equation for P-waves

$$\nabla^2 \phi = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2}$$

Checking irrotationality

$$\nabla \times \mathbf{u}_P = \nabla \times (\nabla \phi) = 0$$

This is always true because the curl of a gradient is identically zero.

Dilatation (volume change)

$$\nabla \cdot \mathbf{u}_P = \nabla \cdot (\nabla \phi) = \nabla^2 \phi \neq 0$$

SV-waves (shear, vertical polarization):

SV-waves are **solenoidal** (divergence-free) and can be described by a vector potential. For 2D motion in the x-z plane, we use:

$$\mathbf{u}_{SV} = \nabla \times (0, \psi, 0) = \nabla \times (\psi \hat{y}) \tag{11}$$

For 2D P-SV case in x-z plane, the vector potential has only a y-component: $\psi = (0, \psi, 0)$.

Detailed derivation

The curl in Cartesian coordinates is:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{12}$$

For $\mathbf{A} = (0, \psi, 0)$:

$$\nabla \times (0, \psi, 0) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \psi & 0 \end{vmatrix} \tag{13}$$

Computing each component:

x-component:

$$(\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial 0}{\partial y} - \frac{\partial \psi}{\partial z} = -\frac{\partial \psi}{\partial z} \tag{14}$$

For 2D problems where nothing depends on y ($\partial/\partial y = 0$):

Standard curl calculation:

$$(\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\frac{\partial \psi}{\partial z} \tag{15}$$

$$(\nabla \times \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0 \tag{16}$$

$$(\nabla \times \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{\partial \psi}{\partial x} \tag{17}$$

$$u_x = -\frac{\partial \psi}{\partial z}, \quad u_z = \frac{\partial \psi}{\partial x} \tag{18}$$

<u>So:</u>

$$\nabla \times \boldsymbol{\psi} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \psi & 0 \end{vmatrix} = -\,\hat{x}\frac{\partial \psi}{\partial z} + \hat{z}\frac{\partial \psi}{\partial x} \tag{19}$$

Therefore:

$$\boxed{u_x = -\frac{\partial \psi}{\partial z}, \quad u_z = +\frac{\partial \psi}{\partial x}} \tag{20}$$

Why this works

Wave equation for SV-waves

$$\nabla^2 \psi = \frac{1}{\beta^2}\frac{\partial^2 \psi}{\partial t^2} \tag{21}$$

Checking solenoidality (divergence-free)

$$\nabla \cdot \mathbf{u}_{SV} = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = \frac{\partial^2 \psi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial z \partial x} = 0 \tag{22}$$

SV-waves involve no volume change.

Rotation (vorticity)

$$\nabla \times \mathbf{u}_{SV} = \nabla \times (\nabla \times (0, \psi, 0)) \neq 0 \tag{23}$$

SV-waves involve **<u>rotation/shear</u>**.

So, from equation 10 and 20:

<u>Horizontal displacement:</u>

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \tag{24}$$

Vertical displacement:

$$u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \tag{25}$$

In matrix form, the continuity of displacement at the interface becomes:

$$\boxed{\begin{pmatrix} u_x \\ u_z \end{pmatrix} = \begin{pmatrix} \partial_x & -\partial_z \\ \partial_z & \partial_x \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}} \tag{26}$$

4. Computing Stress Components in Terms of Potentials

General Stress-Strain Relations

For an isotropic elastic medium, the stress tensor is:

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \tag{27}$$

where:

- λ, μ are Lamé parameters
- δ_{ij} is the Kronecker delta
- $\epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$ is the dilatation (volumetric strain)
- $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the strain tensor

Step 1: Express Dilatation in Terms of Displacement

The dilatation is:

$$\epsilon_{kk} = \nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{28}$$

For 2D problems in the x-z plane (no y-dependence):

$$\epsilon_{kk} = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \tag{29}$$

Step 2: Express Displacements in Terms of Potentials

Recall:

- P-wave: $\mathbf{u}_P = \nabla \phi$, so $u_x = \frac{\partial \phi}{\partial x}$, $u_z = \frac{\partial \phi}{\partial z}$
- SV-wave: $\mathbf{u}_{SV} = \nabla \times (0, \psi, 0)$, so $u_x = -\frac{\partial \psi}{\partial z}$, $u_z = \frac{\partial \psi}{\partial x}$

Total displacement:

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \tag{30}$$
$$u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \tag{31}$$

Step 3: Calculate Dilatation

$$\epsilon_{kk} = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \tag{32}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}\right) \tag{33}$$

$$= \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial z} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial z \partial x} \tag{34}$$

$$= \frac{\partial^2 \phi}{\partial x^2} - \cancel{\frac{\partial^2 \psi}{\partial x \partial z}} + \frac{\partial^2 \phi}{\partial z^2} + \cancel{\frac{\partial^2 \psi}{\partial z \partial x}} \tag{35}$$

$$\epsilon_{kk} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi \tag{36}$$

So:

$$\boxed{\epsilon_{kk} = \nabla \cdot \mathbf{u} = \nabla^2 \phi} \tag{37}$$

This shows that the dilatation depends only on the P-wave potential ϕ, not on the SV-wave potential ψ.

Step 4: Derive σ_{zz}

From the general formula (equation 31):

$$\sigma_{zz} = \lambda \epsilon_{kk} + 2\mu \epsilon_{zz} \tag{38}$$

where:

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{39}$$

Computing ϵ_{zz}

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z} \tag{40}$$

Substituting into σ_{zz}

$$\sigma_{zz} = \lambda \nabla^2 \phi + 2\mu \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right) \tag{41}$$

$$= \lambda \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + 2\mu \frac{\partial^2 \phi}{\partial z^2} + 2\mu \frac{\partial^2 \psi}{\partial x \partial z} \tag{42}$$

$$= \lambda \frac{\partial^2 \phi}{\partial x^2} + \lambda \frac{\partial^2 \phi}{\partial z^2} + 2\mu \frac{\partial^2 \phi}{\partial z^2} + 2\mu \frac{\partial^2 \psi}{\partial x \partial z} \tag{43}$$

$$= \lambda \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + 2\mu \frac{\partial^2 \phi}{\partial z^2} + 2\mu \frac{\partial^2 \psi}{\partial x \partial z} \tag{44}$$

Note that $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}$, So, this can be rewritten as:

$$\sigma_{zz} = \lambda \nabla^2 \phi + 2\mu \frac{\partial^2 \phi}{\partial z^2} + 2\mu \frac{\partial^2 \psi}{\partial x \partial z} \tag{45}$$

$$\boxed{\sigma_{zz} = \lambda \nabla^2 \phi + 2\mu \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right)} \tag{46}$$

Step 5: Derive σ_{xz}

The shear stress is:

$$\sigma_{xz} = 2\mu \epsilon_{xz} = 2 \cdot \mu \cdot \frac{1}{2} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \tag{47}$$

Computing Each Term

First term:

$$\frac{\partial u_x}{\partial z} = \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \right) = \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial z^2} \tag{48}$$

Second term:

$$\frac{\partial u_z}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \right) = \frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \psi}{\partial x^2} \tag{49}$$

Adding the Terms

$$\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} \tag{50}$$

$$= 2 \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x^2} \tag{51}$$

Final Expression for σ_{xz}

$$\sigma_{xz} = \mu \left(2\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x^2} \right) \tag{52}$$

$$\boxed{\sigma_{xz} = \mu \left(2\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} \right)} \tag{53}$$

The fundamental relationships are:

$$\boxed{\epsilon_{kk} = \nabla \cdot \mathbf{u} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \nabla^2 \phi} \tag{54}$$

$$\boxed{\sigma_{zz} = \lambda \nabla^2 \phi + 2\mu \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right) = \lambda(\partial_{xx}\phi + \partial_{zz}\phi) + 2\mu(\partial_{zz}\phi + \partial_{xz}\psi)} \tag{55}$$

$$\boxed{\sigma_{xz} = \mu \left(2\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} \right) = \mu \left(2\partial_{xz}\phi + \partial_{xx}\psi - \partial_{zz}\psi \right)} \tag{56}$$

Physical Interpretation

- The dilatation ϵ_{kk} depends only on the P-wave potential ϕ, confirming that only P-waves cause volume changes.
- The normal stress σ_{zz} involves both ϕ and ψ, showing coupling between P and SV waves at boundaries.
- The shear stress σ_{xz} also involves both potentials, demonstrating mode conversion.
- These expressions allow us to write boundary conditions entirely in terms of ϕ and ψ, which is essential for deriving the Zoeppritz equations.

5. Boundary Conditions at z=0

At an interface (say $z = 0$), we require:

Four Boundary Conditions:

1. Continuity of horizontal displacement: $u_x^{(1)} = u_x^{(2)}$
2. Continuity of vertical displacement: $u_z^{(1)} = u_z^{(2)}$
3. Continuity of shear stress: $\sigma_{xz}^{(1)} = \sigma_{xz}^{(2)}$
4. Continuity of normal stress: $\sigma_{zz}^{(1)} = \sigma_{zz}^{(2)}$

At a free surface ($z = 0$), we require:

1. $\sigma_{xz} = 0$
2. $\sigma_{zz} = 0$

These conditions, expressed in terms of ϕ and ψ using the formulas derived above, lead to the famous Zoeppritz equations for reflection and transmission coefficients. These four equations also determine the four unknowns: A', B_1, A_2, B_2.

6. Detailed Derivation of Boundary Conditions

Computing Derivatives

For $\phi_1 = A_1 e^{i(\omega t - k_x x + k_{z\alpha_1} z)} + A' e^{i(\omega t - k_x x - k_{z\alpha_1} z)}$ at $z = 0$:

$$\left.\frac{\partial \phi_1}{\partial x}\right|_{z=0} = -ik_x (A_1 + A') e^{i(\omega t - k_x x)} \tag{57}$$

$$\left.\frac{\partial \phi_1}{\partial z}\right|_{z=0} = ik_{z\alpha_1} (A_1 - A') e^{i(\omega t - k_x x)} \tag{58}$$

For $\psi_1 = B_1 e^{i(\omega t - k_x x - k_{z\beta_1} z)}$ at $z = 0$:

$$\left.\frac{\partial \psi_1}{\partial x}\right|_{z=0} = -ik_x B_1 e^{i(\omega t - k_x x)} \tag{59}$$

$$\left.\frac{\partial \psi_1}{\partial z}\right|_{z=0} = -ik_{z\beta_1} B_1 e^{i(\omega t - k_x x)} \tag{60}$$

Similarly for medium 2 at $z = 0$:

$$\phi_2 = A_2 e^{i(\omega t - k_x x + k_{z\alpha_2} z)} \tag{61}$$

$$\psi_2 = B_2 e^{i(\omega t - k_x x + k_{z\beta_2} z)} \tag{62}$$

$$\left.\frac{\partial \phi_2}{\partial x}\right|_{z=0} = -ik_x A_2 e^{i(\omega t - k_x x)} \tag{63}$$

$$\left.\frac{\partial \phi_2}{\partial z}\right|_{z=0} = ik_{z\alpha_2} A_2 e^{i(\omega t - k_x x)} \tag{64}$$

$$\left.\frac{\partial \psi_2}{\partial x}\right|_{z=0} = -ik_x B_2 e^{i(\omega t - k_x x)} \tag{65}$$

$$\left.\frac{\partial \psi_2}{\partial z}\right|_{z=0} = ik_{z\beta_2} B_2 e^{i(\omega t - k_x x)} \tag{66}$$

Boundary Condition 1: Horizontal Displacement

$$u_x^{(1)}\big|_{z=0} = u_x^{(2)}\big|_{z=0} \tag{67}$$

$$\left.\frac{\partial \phi_1}{\partial x} - \frac{\partial \psi_1}{\partial z}\right|_{z=0} = \left.\frac{\partial \phi_2}{\partial x} - \frac{\partial \psi_2}{\partial z}\right|_{z=0} \tag{68}$$

$$-ik_x(A_1 + A') - (-ik_{z\beta_1} B_1) = -ik_x A_2 - ik_{z\beta_2} B_2 \tag{69}$$

Simplifying:

$$k_x(A_1 + A') - k_{z\beta_1} B_1 = k_x A_2 + k_{z\beta_2} B_2 \tag{70}$$

Rearrange:

$$\boxed{k_x\, A' - k_{z\beta_1} B_1 - k_x A_2 - k_{z\beta_2} B_2 = -k_x\, A_1}$$

Boundary Condition 2: Vertical Displacement

$$u_z^{(1)}\big|_{z=0} = u_z^{(2)}\big|_{z=0} \tag{71}$$

$$\left.\frac{\partial \phi_1}{\partial z} + \frac{\partial \psi_1}{\partial x}\right|_{z=0} = \left.\frac{\partial \phi_2}{\partial z} + \frac{\partial \psi_2}{\partial x}\right|_{z=0} \tag{72}$$

$$ik_{z\alpha_1}(A_1 - A') + (-ik_x B_1) = ik_{z\alpha_2} A_2 + (-ik_x B_2) \tag{73}$$

Simplifying:

$$k_{z\alpha_1}(A_1 - A') - k_x B_1 = k_{z\alpha_2} A_2 - k_x B_2 \tag{74}$$

$$\boxed{-k_{z\alpha_1} A' - k_x B_1 - k_{z\alpha_2} A_2 + k_x B_2 = -k_{z\alpha_1} A_1}$$

Boundary Condition 3: Shear Stress

$$\sigma_{xz}^{(1)}\big|_{z=0} = \sigma_{xz}^{(2)}\big|_{z=0} \tag{75}$$

After computing second derivatives:

$$\mu_1[2\partial_{xz}\phi_1 + \partial_{xx}\psi_1 - \partial_{zz}\psi_1]_{z=0} = \mu_2[2\partial_{xz}\phi_2 + \partial_{xx}\psi_2 - \partial_{zz}\psi_2]_{z=0} \tag{76}$$

Computing the second and mixed derivatives using equation 59 - 68:

$$\partial_{xz}\phi_1|_{z=0} = (-ik_x)(ik_{z\alpha_1})(A_1 - A')e^{i(\omega t - k_x x)} = k_x k_{z\alpha_1}(A_1 - A')e^{i(\omega t - k_x x)} \tag{77}$$

$$\partial_{xx}\psi_1|_{z=0} = (-ik_x)^2 B_1 e^{i(\omega t - k_x x)} = -k_x^2 B_1 e^{i(\omega t - k_x x)} \tag{78}$$

$$\partial_{zz}\psi_1|_{z=0} = (-ik_{z\beta_1})^2 B_1 e^{i(\omega t - k_x x)} = -k_{z\beta_1}^2 B_1 e^{i(\omega t - k_x x)} \tag{79}$$

$$\partial_{xz}\phi_2|_{z=0} = (-ik_x)(ik_{z\alpha_2})\, A_2\, e^{i(\omega t - k_x x)} = k_x k_{z\alpha_2}\, A_2\, e^{i(\omega t - k_x x)} \tag{80}$$

$$\partial_{xx}\psi_2|_{z=0} = (-ik_x)^2 B_2 e^{i(\omega t - k_x x)} = -k_x^2 B_2 e^{i(\omega t - k_x x)} \tag{81}$$

$$\partial_{zz}\psi_2|_{z=0} = (ik_{z\beta_2})^2 B_2 e^{i(\omega t - k_x x)} = -k_{z\beta_2}^2 B_2 e^{i(\omega t - k_x x)} \tag{82}$$

Substitute the derivatives and cancel the common terms, this yields:

$$\boxed{\mu_1[2k_x k_{z\alpha_1}(A_1 - A') + (k_{z\beta_1}^2 - k_x^2)B_1] = \mu_2[2k_x k_{z\alpha_2}A_2 + (k_{z\beta_2}^2 - k_x^2)B_2]} \tag{83}$$

$$-2\mu_1 k_x k_{z\alpha_1}A' + \mu_1(k_{z\beta_1}^2 - k_x^2)B_1 - 2\mu_2 k_x k_{z\alpha_2}A_2 - \mu_2(k_{z\beta_2}^2 - k_x^2)B_2 = -2\mu_1 k_x k_{z\alpha_1}A_1$$

Multiply both sides by -1:

$$\boxed{2\mu_1 k_x k_{z\alpha_1}A' + \mu_1(k_x^2 - k_{z\beta_1}^2)B_1 + 2\mu_2 k_x k_{z\alpha_2}A_2 - \mu_2(k_x^2 - k_{z\beta_2}^2)B_2 = 2\mu_1 k_x k_{z\alpha_1}A_1}$$

Boundary Condition 4: Normal Stress

$$\sigma_{zz1}\big|_{z=0} = \sigma_{zz2}\big|_{z=0} \tag{84}$$

$$\begin{aligned}\lambda_1(\partial_{xx}\phi_1 + \partial_{zz}\phi_1) + 2\mu_1(\partial_{zz}\phi_1 + \partial_{xz}\psi_1)\big|_{z=0}& \\ = \lambda_2(\partial_{xx}\phi_2 + \partial_{zz}\phi_2) + 2\mu_2(\partial_{zz}\phi_2 + \partial_{xz}\psi_2)\big|_{z=0}&\end{aligned} \tag{85}$$

Computing:

$$\partial_{xx}\phi_1|_{z=0} = -k_x^2(A_1 + A')e^{i(\omega t - k_x x)} \tag{86}$$
$$\partial_{zz}\phi_1|_{z=0} = -k_{z\alpha_1}^2(A_1 + A')e^{i(\omega t - k_x x)} \tag{87}$$
$$\partial_{xz}\psi_1|_{z=0} = -k_x k_{z\beta_1} B_1 e^{i(\omega t - k_x x)} \tag{88}$$

$$\partial_{xx}\phi_2|_{z=0} = -k_x^2\, A_2\, e^{i(\omega t - k_x x)} \tag{89}$$
$$\partial_{zz}\phi_2|_{z=0} = -k_{z\alpha_2}^2\, A_2\, e^{i(\omega t - k_x x)} \tag{90}$$
$$\partial_{xz}\psi_2|_{z=0} = k_x k_{z\beta_2} B_2 e^{i(\omega t - k_x x)} \tag{91}$$

After substitution and simplification:

$$\begin{aligned}\lambda_1(-k_x^2 - k_{z\alpha_1}^2)(A_1 + A') + 2\mu_1(-k_{z\alpha_1}^2)(A_1 + A') - 2\mu_1 k_x k_{z\beta_1} B_1 \\ = \lambda_2(-k_x^2 - k_{z\alpha_2}^2)A_2 + 2\mu_2(-k_{z\alpha_2}^2)A_2 + 2\mu_2 k_x k_{z\beta_2} B_2\end{aligned} \tag{92}$$

This can be rewritten as:

$$\begin{aligned}\left[\lambda_1(k_x^2 + k_{z\alpha_1}^2) + 2\mu_1 k_{z\alpha_1}^2\right](A_1 + A') + 2\mu_1 k_x k_{z\beta_1} B_1 \\ = [\lambda_2(k_x^2 + k_{z\alpha_2}^2) + 2\mu_2 k_{z\alpha_2}^2]A_2 - 2\mu_2 k_x k_{z\beta_2} B_2\end{aligned} \tag{93}$$

$$\begin{aligned}\left[\lambda_1(k_x^2 + k_{z\alpha_1}^2) + 2\mu_1 k_{z\alpha_1}^2\right]A' + 2\mu_1 k_x k_{z\beta_1} B_1 - \left[\lambda_2(k_x^2 + k_{z\alpha_2}^2) + 2\mu_2 k_{z\alpha_2}^2\right]A_2 + 2\mu_2 k_x k_{z\beta_2} B_2 \\ = -\left[\lambda_1(k_x^2 + k_{z\alpha_1}^2) + 2\mu_1 k_{z\alpha_1}^2\right]A_1\end{aligned} \tag{94}$$

Let:

$$M_1 = \lambda_1(k_x^2 + k_{z\alpha_1}^2) + 2\mu_1 k_{z\alpha_1}^2$$

$$M_2 = \lambda_2(k_x^2 + k_{z\alpha_2}^2) + 2\mu_2 k_{z\alpha_2}^2$$

$$\boxed{M_1 A' + 2\mu_1 k_x k_{z\beta_1} B_1 - M_2 A_2 + 2\mu_2 k_x k_{z\beta_2} B_2 = -M_1 A_1}$$

Matrix Form (Zoeppritz System)

Collecting the four equations from the boundary conditions into matrix form, $\mathbf{M}\,x = \mathbf{b}$, where $\mathbf{b}$ is the right-hand vector containing the incident-wave terms (proportional to A_1). Explicitly the coefficient matrix (rows are eqns from the BCs, columns unknowns $[A', B_1, A_2, B_2]$) is:

$$\begin{pmatrix} k_x & -k_{z\beta 1} & -k_x & -k_{z\beta 2} \\ -k_{z\alpha 1} & -k_x & -k_{z\alpha 2} & k_x \\ 2\mu_1 k_x k_{z\alpha 1} & \mu_1(k_x^2 - k_{z\beta_1}^2) & 2\mu_2 k_x k_{z\alpha 2} & -\mu_2(k_x^2 - k_{z\beta_2}^2) \\ M_1 & 2\mu_1 k_x k_{z\beta 1} & -M_2 & 2\mu_2 k_x k_{z\beta 2} \end{pmatrix} \begin{pmatrix} A' \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} -k_x A_1 \\ -k_{z\alpha 1} A_1 \\ 2\mu_1 k_x k_{z\alpha 1} A_1 \\ -M_1 A_1 \end{pmatrix}$$

Divide the entire system by A_1:

$$\boxed{\begin{pmatrix} k_x & -k_{z\beta 1} & -k_x & -k_{z\beta 2} \\ -k_{z\alpha 1} & -k_x & -k_{z\alpha 2} & k_x \\ 2\mu_1 k_x k_{z\alpha 1} & \mu_1(k_x^2 - k_{z\beta_1}^2) & 2\mu_2 k_x k_{z\alpha 2} & -\mu_2(k_x^2 - k_{z\beta_2}^2) \\ M_1 & 2\mu_1 k_x k_{z\beta 1} & -M_2 & 2\mu_2 k_x k_{z\beta 2} \end{pmatrix} \begin{pmatrix} R_{PP} \\ R_{PS} \\ T_{PP} \\ T_{PS} \end{pmatrix} = \begin{pmatrix} -k_x \\ -k_{z\alpha 1} \\ 2\mu_1 k_x k_{z\alpha 1} \\ -M_1 \end{pmatrix}}$$

Where:

$$M_1 = \lambda_1(k_x^2 + k_{z\alpha_1}^2) + 2\mu_1 k_{z\alpha_1}^2$$

$$M_2 = \lambda_2(k_x^2 + k_{z\alpha_2}^2) + 2\mu_2 k_{z\alpha_2}^2$$

The reflection and transmission coefficients are therefore:

$$R_{PP} = \frac{A'}{A_1}, \qquad R_{PS} = \frac{B_1}{A_1}, \qquad T_{PP} = \frac{A_2}{A_1}, \qquad T_{PS} = \frac{B_2}{A_1}.$$

This 4×4 linear system is the general **Zoeppritz equation** for a P-wave incident on a plane elastic interface. Solving for RC and TC gives:

$$\begin{pmatrix} R_{PP} \\ R_{PS} \\ T_{PP} \\ T_{PS} \end{pmatrix} = \mathbf{M}^{-1}\mathbf{b}.$$

Where:

$$\mathbf{M} = \begin{pmatrix} k_x & -k_{z\beta 1} & -k_x & -k_{z\beta 2} \\ -k_{z\alpha 1} & -k_x & -k_{z\alpha 2} & k_x \\ 2\mu_1 k_x k_{z\alpha 1} & \mu_1(k_x^2 - k_{z\beta_1}^2) & 2\mu_2 k_x k_{z\alpha 2} & -\mu_2(k_x^2 - k_{z\beta_2}^2) \\ M_1 & 2\mu_1 k_x k_{z\beta 1} & -M_2 & 2\mu_2 k_x k_{z\beta 2} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} -k_x \\ -k_{z\alpha 1} \\ 2\mu_1 k_x k_{z\alpha 1} \\ -M_1 \end{pmatrix}$$

The Zoeppritz system of equation above in its potential-based form can be transformed into a more compact form, similar to the one presented in the classical book by Aki and Richards

7. Special Case: P-Sv R/T Coefficients at Free Surface (Earth's Surface)

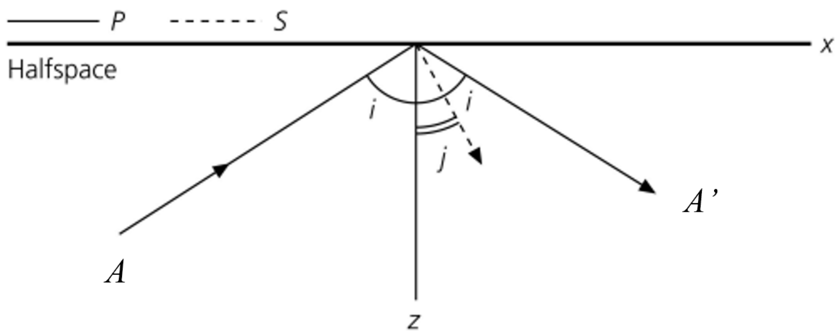

Figure 2: P-SV R/T Coefficients at Earth Surface

The free surface (e.g., the Earth's surface) is a special case where Medium 2 is a vacuum. This simplifies the problem because there are no transmitted waves and the boundary conditions become **stress-free**.

At Earth's surface, the boundary conditions simplify to **stress-free conditions**:

$$\sigma_{xz}\big|_{z=0} = 0, \quad \sigma_{zz}\big|_{z=0} = 0 \tag{95}$$

This gives us two equations for two unknowns (A', B_1):

Free Surface Boundary Conditions

- Medium 1: Elastic solid with parameters $\alpha, \beta, \rho, \lambda, \mu$
- Medium 2: Vacuum ($z > 0$)
- Boundary conditions at $z = 0$: $\sigma_{zz} = 0$ and $\sigma_{xz} = 0$

Shear Stress = 0

From **Boundary Condition 3** with medium 2 absent:

$$\mu[2k_x k_{z\alpha}(A_1 - A') + (k_{z\beta}^2 - k_x^2)B_1] \tag{96}$$

$$2k_x k_{z\alpha}(A_1 - A') + (k_x^2 - k_{z\beta}^2)B_1 = 0 \tag{97}$$

Dividing by k_x^2:

$$2\frac{k_{z\alpha}}{k_x}(A_1 - A') + \left(\frac{k_{z\beta}^2}{k_x^2} - 1\right) B_1 = 0 \tag{98}$$

Defining:

$$\boxed{r_\alpha = \frac{k_{z\alpha}}{k_x} = \cot i_\alpha \quad \text{and} \quad r_\beta = \frac{k_{z\beta}}{k_x} = \cot i_\beta}$$

Note that $\frac{k_{z\beta}^2}{k_x^2} = r_\beta^2$, so:

$$\boxed{2r_\alpha(A_1 - A') + (r_\beta^2 - 1)B_1 = 0} \tag{99}$$

Normal Stress = 0

From **Boundary Condition 4** with medium 2 absent:

$$\lambda(k_x^2 + k_{z\alpha}^2)(A_1 + A') + 2\mu(k_{z\alpha}^2)(A_1 + A') + 2\mu k_x k_{z\beta} B_1 = 0$$

$$[\lambda(k_x^2 + k_{z\alpha}^2) + 2\mu k_{z\alpha}^2](A_1 + A') + 2\mu k_x k_{z\beta} B_1 = 0 \tag{100}$$

We can rewrite:

$$[\lambda(k_x^2 + k_{z\alpha}^2) + 2\mu k_{z\alpha}^2](A_1 + A') = -2\mu k_x k_{z\beta} B_1 \tag{101}$$

Dividing by k_x^2:

$$\left[\lambda\left(1+\frac{k_{z\alpha}^2}{k_x^2}\right)+2\mu\frac{k_{z\alpha}^2}{k_x^2}\right](A_1+A')=-2\mu\frac{k_{z\beta}}{k_x}B_1 \tag{102}$$

$$[\lambda(1+r_\alpha^2)+2\mu r_\alpha^2](A_1+A')=-2\mu r_\beta B_1 \tag{103}$$

This can be rewritten as:

$$\lambda(1+r_\alpha^2)(A_1+A')+2\mu r_\alpha^2(A_1+A')+2\mu r_\beta B_1=0$$

$$\boxed{\lambda(1+r_\alpha^2)(A_1+A')+2\mu\left(r_\alpha^2(A_1+A')+r_\beta B_1\right)=0} \tag{104}$$

Solving for Reflection Coefficients

System of Equations

From Equation 99:

$$2r_\alpha(A_1-A')+(r_\beta^2-1)B_1=0$$

$$\boxed{A_1-A'=-\frac{(r_\beta^2-1)}{2r_\alpha}B_1=\frac{(1-r_\beta^2)}{2r_\alpha}B_1} \tag{105}$$

From Equation 104:

$$\lambda(1+r_\alpha^2)(A_1+A')+2\mu\left(r_\alpha^2(A_1+A')+r_\beta B_1\right)=0$$

$$\left[\lambda(1+r_\alpha^2)+2\mu r_\alpha^2\right](A_1+A')=-2\mu r_\beta B_1$$

$$\boxed{A_1+A'=\frac{-2\mu r_\beta B_1}{\lambda(1+r_\alpha^2)+2\mu r_\alpha^2}} \tag{106}$$

Solving by Elimination Method

Let:

$$S=A_1+A' \quad \text{(sum)} \tag{107}$$

$$D=A_1-A' \quad \text{(difference)} \tag{108}$$

Then: $A_1=(S+D)/2$ and $A'=(S-D)/2$

From Eq. 105:

$$D = \frac{(1 - r_\beta^2)}{2r_\alpha} B_1 \tag{109}$$

From Eq. 106:

$$S = -\frac{2\mu r_\beta}{\lambda(1 + r_\alpha^2) + 2\mu r_\alpha^2} B_1 \tag{110}$$

Therefore:

$$A_1 = \frac{1}{2}\left[\frac{(1 - r_\beta^2)}{2r_\alpha} - \frac{2\mu r_\beta}{\lambda(1 + r_\alpha^2) + 2\mu r_\alpha^2}\right] B_1 \tag{111}$$

$$A' = \frac{1}{2}\left[-\frac{2\mu r_\beta}{\lambda(1 + r_\alpha^2) + 2\mu r_\alpha^2} - \frac{(1 - r_\beta^2)}{2r_\alpha}\right] B_1 \tag{112}$$

Final Expressions

After significant algebraic manipulation, the reflection coefficients are:

P-to-P Reflection Coefficient:

$$\boxed{R_P = \frac{A'}{A_1} = \frac{4r_\alpha r_\beta - (r_\beta^2 - 1)^2}{4r_\alpha r_\beta + (r_\beta^2 - 1)^2}} \tag{113}$$

P-to-SV Conversion Coefficient:

$$\boxed{R_{SV} = \frac{B_1}{A_1} = \frac{4r_\alpha(1 - r_\beta^2)}{4r_\alpha r_\beta + (r_\beta^2 - 1)^2}} \tag{114}$$

where:

$$r_\alpha = \frac{k_{z\alpha}}{k_x} = \cot i_\alpha = \frac{\cos i_\alpha}{\sin i_\alpha} \tag{115}$$

$$r_\beta = \frac{k_{z\beta}}{k_x} = \cot i_\beta = \frac{\cos i_\beta}{\sin i_\beta} \tag{116}$$

Note: The angles i_α and i_β are measured from the vertical (normal to the surface).

Remember the relations:

- $\lambda + 2\mu = \rho\alpha^2$
- $\mu = \rho\beta^2$
- $k_x^2 + k_{z\alpha}^2 = \omega^2/\alpha^2 = k_\alpha^2$

8. Physical Interpretation

Key Features

1. **Mode Conversion**: Unlike SH waves, P-waves generate both P and SV reflected waves at a free surface.

2. **Critical Angles**: When $i > i_c = \sin^{-1}(\alpha/\beta)$, the SV wave becomes inhomogeneous (evanescent).

3. **Amplitude Behavior**:

   - At normal incidence ($i = 0$): $r_\alpha \to \infty$, $r_\beta \to \infty$, giving $R_P = -1$, $R_{SV} = 0$
   - At grazing incidence ($i \to 90$): $r_\alpha \to 0$, $r_\beta \to 0$, giving $R_P \to 1$, $R_{SV} \to 0$
   - Maximum P-to-SV conversion occurs at intermediate angles

4. **Energy Conservation**:

$$|R_P|^2 + |R_{SV}|^2 \frac{I_\beta \cos j_\beta}{I_\alpha \cos i_\alpha} = 1 \tag{117}$$

   where $I = \rho v$ is the impedance.

Special Values

At normal incidence ($i_\alpha = 0$), using L'Hôpital's rule or direct evaluation:

$$R_P(0) = -1 \quad \text{(phase reversal)} \tag{118}$$
$$R_{SV}(0) = 0 \quad \text{(no mode conversion)} \tag{119}$$

At the critical angle $i_c = \sin^{-1}(\alpha/\beta)$:

- The SV wave grazes along the surface ($i_\beta = 90$)
- $r_\beta = 0$
- Maximum mode conversion occurs

9. Shuey's Approximation (Linearized AVO)

For small contrasts across an interface and weak angles, the P-wave reflection coefficient can be approximated:

$$\boxed{R(\theta) = R(0) + G\sin^2\theta + F(\tan^2\theta - \sin^2\theta)} \tag{120}$$

where:

Intercept (normal incidence reflectivity):

$$R(0) = \frac{1}{2}\left(\frac{\Delta\rho}{\rho} + \frac{\Delta\alpha}{\alpha}\right) \tag{121}$$

Gradient (AVO slope):

$$G = \frac{\Delta\alpha}{2\alpha} - \frac{2\beta^2}{\alpha}\left(\frac{\Delta\rho}{\rho} + \frac{2\Delta\beta}{\beta}\right) \tag{122}$$

Large-angle term:

$$F = \frac{\Delta\alpha}{2\alpha} \tag{123}$$

Here:

- $\Delta\rho = \rho_2 - \rho_1$, $\Delta\alpha = \alpha_2 - \alpha_1$, $\Delta\beta = \beta_2 - \beta_1$
- $\rho = (\rho_1 + \rho_2)/2$, $\alpha = (\alpha_1 + \alpha_2)/2$, $\beta = (\beta_1 + \beta_2)/2$
- θ is the angle of incidence

This approximation is widely used in seismic exploration for amplitude-versus-offset (AVO) analysis to detect hydrocarbons and characterize lithology.

Physical Meaning of AVO Parameters

- $R(0)$: Measures the impedance contrast at normal incidence
- G: Captures the angle-dependent behavior, sensitive to fluid content and Poisson's ratio changes
- F: Important at large angles, related to P-wave velocity contrast

Summary

The P-SV problem is significantly more complex than the SH problem due to:

1. Mode conversion between P and SV waves
2. Coupling through both displacement and stress boundary conditions
3. Four coupled equations instead of two
4. Angle-dependent behavior that depends on both α and β

The exact Zoeppritz equations provide the complete solution, while Shuey's approximation offers a practical, linearized alternative for seismic interpretation.