Navier-Cauchy equation in linear elasticity using the displacement formulation.

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The Navier-Cauchy equation is the fundamental partial differential equation of motion in linear elasticity theory. It describes how elastic solids deform and move under the influence of internal and external forces. Named after Claude-Louis Navier (1785 - 1836) and Augustin-Louis Cauchy (1789 - 1857), it is derived from Newton's second law and Hooke's law for elastic solids.

General Form of the Navier-Cauchy Equation

For an isotropic, homogeneous elastic solid with small deformations, the Navier-Cauchy equation in displacement form is:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mathbf{f} \tag{1}$$

Where:

- $\mathbf{u}(\mathbf{x})$ = displacement vector field
- λ, μ = Lamé parameters (elastic constants)
- $\mathbf{F}$ = body force per unit volume (e.g., gravity)
- ρ = mass density of the material
- ∇^2 = Vector Laplacian (applied component-wise).
- $\nabla \cdot \mathbf{u}$ = divergence of displacement (volumetric strain)

Static Case (Equilibrium Equation)

If the system is in static equilibrium (no time dependence), the equation simplifies to:

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mathbf{F} = \mathbf{0}$$

This equation describes the elastostatics, the deformation of a solid at rest.

Interpretation

- The first term ($\mu\nabla^2\mathbf{u}$) represents shear deformation.
- The second term ($(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u})$) accounts for volumetric deformation.
- $\mathbf{F}$ includes external forces like gravity.

Alternative Form (Using Young's Modulus & Poisson's Ratio)

If the material properties are given in terms of **Young's modulus** E and **Poisson's ratio** ν, the Lamé parameters are:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

Applications

- Used in solid mechanics to model stress and strain in structures.
- Helps solve problems in **geophysics** (e.g., seismic wave propagation).
- Essential in **engineering** (e.g., analysing deformations in beams, plates, and shells).

Special Cases

1. **Incompressible Material** ($\nabla \cdot \mathbf{u} = 0$):

$$\nabla^2\mathbf{u} + \frac{\mathbf{F}}{\mu} = \mathbf{0}$$

   What do you notice? The equation resembles the Poisson equation.

2. **Irrotational Deformation** ($\nabla \times \mathbf{u} = \mathbf{0}$):

$$(\lambda + 2\mu)\nabla^2 \mathbf{u} + \mathbf{F} = \mathbf{0}$$

(Leads to a wave equation in dynamic cases.)

Now, we are going to derive the Navier-Cauchy equation using the Indicial or index notation.

Derivation of Navier-Cauchy Equation

Step 1: Newton's Second Law (Conservation of Linear Momentum)

For a continuous medium, Newton's second law says:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}$$

Where:

- ρ: mass density
- $\mathbf{u}$: displacement vector
- $\boldsymbol{\sigma}$: stress tensor
- $\mathbf{f}$: body force per unit volume

This equation equates the inertial force (LHS) to the sum of internal (stress divergence) and external (body force) forces (RHS).

Step 2: Hooke's Law (Stress-Strain Relationship for Isotropic Materials)

For a linear, isotropic, and homogeneous solid, Hooke's law gives:

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}$$

Where:

- λ and μ: Lamé constants
- δ_{ij}: Kronecker delta
- $\epsilon_{kk} = \nabla \cdot \mathbf{u}$: trace of strain tensor (volumetric strain)

- $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$: linearized strain tensor

Step 3: Compute $\nabla \cdot \boldsymbol{\sigma}$ (Stress Divergence)

We substitute Hooke's law into the divergence:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(\lambda \delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij}) = \lambda \frac{\partial}{\partial x_i}\epsilon_{kk} + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Simplify:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \lambda \frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{u}) + \mu \nabla^2 u_i + \mu \frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{u})$$

Group terms:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \mu \nabla^2 u_i + (\lambda + \mu)\frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{u})$$

Step 4: Substitute into Newton's Second Law

Now substitute this into the momentum equation:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \mu \nabla^2 u_i + (\lambda + \mu)\frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{u}) + f_i$$

Or in vector form:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mathbf{f}$$

✓ This is the Navier-Cauchy equation in displacement form for linear isotropic elasticity.

Now, we will derive the elastostatics equation when there's no inertia force (zero-time-dependency). The derivation is similar to the above but will be in vector or direct notation.

Derivation of Equilibrium Equation in Vector Notation

1. Strain-Displacement Relationship (Kinematics)

The infinitesimal strain tensor $\boldsymbol{\epsilon}$ is defined in terms of displacement gradients:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\text{or in direct notation:} \quad \boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$$

2. Stress-Strain Relationship (Constitutive Law)

For a linear isotropic elastic material, the stress tensor $\boldsymbol{\sigma}$ is related to strain via Hooke's law:

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

where:

- $\lambda, \mu =$ Lamé parameters
- $\epsilon_{kk} = \nabla \cdot \mathbf{u}$ (volumetric strain)
- $\delta_{ij} =$ Kronecker delta

$$\text{In direct notation:} \quad \boldsymbol{\sigma} = \lambda(\nabla \cdot \mathbf{u})\mathbf{I} + 2\mu\boldsymbol{\epsilon}$$

3. Equilibrium Equation (Balance of Forces)

The static equilibrium condition (Newton's second law) in the absence of inertial effects is:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{F} = \mathbf{0}$$

where $\mathbf{F}$ is the body force per unit volume (e.g., gravity). Substituting the stress-strain relationship:

$$\nabla \cdot [\lambda(\nabla \cdot \mathbf{u})\mathbf{I} + 2\mu\boldsymbol{\epsilon}] + \mathbf{F} = \mathbf{0}$$

4. Expanding the Divergence of Stress

The divergence of $\boldsymbol{\sigma}$ can be expanded as:

$$\nabla \cdot \boldsymbol{\sigma} = \nabla(\lambda\nabla \cdot \mathbf{u}) + 2\mu\nabla \cdot \boldsymbol{\epsilon}$$

Since $\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$, we have:

$$\nabla \cdot \boldsymbol{\epsilon} = \frac{1}{2}\nabla \cdot (\nabla\mathbf{u}) + \frac{1}{2}\nabla(\nabla \cdot \mathbf{u}) = \frac{1}{2}\nabla^2\mathbf{u} + \frac{1}{2}\nabla(\nabla \cdot \mathbf{u})$$

Thus:

$$\nabla \cdot \boldsymbol{\sigma} = \lambda\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2\mathbf{u} + \mu\nabla(\nabla \cdot \mathbf{u})$$

5. Combining Terms

Substituting back into the equilibrium equation:

$$\lambda\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2\mathbf{u} + \mu\nabla(\nabla \cdot \mathbf{u}) + \mathbf{F} = \mathbf{0}$$

Grouping terms:

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mathbf{F} = \mathbf{0}$$

This is the Navier-Cauchy equation for static elasticity

Wave Decomposition in Elastic Solids

Now, let's see how the Navier-Cauchy equation separates into P-wave (compressional) and S-wave (shear) components. This decomposition is done using Helmholtz decomposition of the displacement field.

1: Navier-Cauchy Equation

Starting from the dynamic Navier-Cauchy equation (body forces neglected, $\mathbf{F} = 0$):

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) \tag{2}$$

2: Helmholtz Decomposition

The displacement field can be decomposed into irrotational (P-wave) and solenoidal (S-wave) components:

$$\mathbf{u} = \nabla \phi + \nabla \times \mathbf{\Psi}$$

where:

- ϕ is the scalar potential (corresponds to P-waves, compressional)
- $\mathbf{\Psi}$ is the vector potential (corresponds to S-waves, shear)
- Key properties:

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{\Psi}) &= 0 \\ \nabla \times (\nabla \phi) &= 0 \end{aligned}$$

3: Acceleration Term – LHS of Equation 2

Starting from the Helmholtz decomposition of the displacement field:

$$\mathbf{u} = \nabla \phi + \nabla \times \mathbf{\Psi}$$

The acceleration term becomes:

$$\begin{aligned}\frac{\partial^2 \mathbf{u}}{\partial t^2} &= \frac{\partial^2}{\partial t^2}\left(\nabla\phi + \nabla\times\mathbf{\Psi}\right)\\ &= \nabla\left(\frac{\partial^2\phi}{\partial t^2}\right) + \nabla\times\left(\frac{\partial^2\mathbf{\Psi}}{\partial t^2}\right)\end{aligned}$$

Thus, the left-hand side (LHS) of the Navier-Cauchy equation transforms as:

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho\nabla\left(\frac{\partial^2\phi}{\partial t^2}\right) + \rho\nabla\times\left(\frac{\partial^2\mathbf{\Psi}}{\partial t^2}\right) \tag{3}$$

4: RHS of Equation 2 – Laplacian and Gradient Terms

The right-hand side (RHS) of the Navier-Cauchy equation contains two key terms:

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})$$

4.1 Vector Laplacian Term

Applying the vector Laplacian to the Helmholtz decomposition:

$$\begin{aligned}\nabla^2\mathbf{u} &= \nabla^2(\nabla\phi + \nabla\times\mathbf{\Psi})\\ &= \nabla(\nabla^2\phi) + \nabla\times(\nabla^2\mathbf{\Psi}) \quad \text{(since } \nabla^2 \text{ commutes with } \nabla \text{ and } \nabla\times\text{)}\end{aligned}$$

Thus:

$$\mu\nabla^2\mathbf{u} = \mu\nabla(\nabla^2\phi) + \mu\nabla\times(\nabla^2\mathbf{\Psi}) \tag{4}$$

4.2 Gradient of Divergence Term

First compute the divergence:

$$\nabla\cdot\mathbf{u} = \nabla\cdot(\nabla\phi) + \nabla\cdot(\nabla\times\mathbf{\Psi}) = \nabla^2\phi + 0 = \nabla^2\phi$$

Then take its gradient:

$$\nabla(\nabla\cdot\mathbf{u}) = \nabla(\nabla^2\phi)$$

Therefore:

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) = (\lambda+\mu)\nabla(\nabla^2\phi) \tag{5}$$

Key Notes:

- The vector Laplacian maintains both ϕ and $\mathbf{\Psi}$ components
- The divergence only affects the scalar potential ϕ (P-waves)
- The solenoidal component $\mathbf{\Psi}$ naturally disappears in the divergence operation

5: Final Substitution into the Wave Equations

Substituting the decomposed terms (3), (4), and (5) back into the Navier-Cauchy equation (2):

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})$$

Remember:

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \rho\nabla\left(\frac{\partial^2\phi}{\partial t^2}\right) + \rho\nabla\times\left(\frac{\partial^2\mathbf{\Psi}}{\partial t^2}\right)$$

$$\mu\nabla^2\mathbf{u} = \mu\nabla(\nabla^2\phi) + \mu\nabla\times(\nabla^2\mathbf{\Psi})$$

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) = (\lambda+\mu)\nabla(\nabla^2\phi)$$

So,

$$\rho\nabla\left(\frac{\partial^2\phi}{\partial t^2}\right) + \rho\nabla\times\left(\frac{\partial^2\mathbf{\Psi}}{\partial t^2}\right) = \mu\nabla(\nabla^2\phi) + \mu\nabla\times(\nabla^2\mathbf{\Psi}) + (\lambda+\mu)\nabla(\nabla^2\phi)$$

Due to commutativity of differential operator,

$$\mu\nabla(\nabla^2\phi) + \mu\nabla\times(\nabla^2\mathbf{\Psi}) = \mu\nabla^2(\nabla\phi) + \mu\nabla^2(\nabla\times\mathbf{\Psi})$$

Then,

$$\rho\left[\nabla\frac{\partial^2\phi}{\partial t^2} + \nabla\times\frac{\partial^2\mathbf{\Psi}}{\partial t^2}\right] = (\lambda+\mu)\nabla(\nabla^2\phi) + \mu\nabla^2(\nabla\phi) + \mu\nabla^2(\nabla\times\mathbf{\Psi})$$

Separation of Components

- **Irrotational (P-wave) components** (all gradient terms):

$$\rho\nabla\frac{\partial^2\phi}{\partial t^2} = (\lambda+\mu)\nabla(\nabla^2\phi) + \mu\nabla(\nabla^2\phi)$$
$$= (\lambda+2\mu)\nabla(\nabla^2\phi)$$

Since $\nabla^2(\nabla\phi) = \nabla(\nabla^2\phi)$, we can factor out ∇:

$$\nabla\left[\rho\frac{\partial^2\phi}{\partial t^2} - (\lambda+2\mu)\nabla^2\phi\right] = 0$$

This implies:

$$\boxed{\rho\frac{\partial^2\phi}{\partial t^2} = (\lambda+2\mu)\nabla^2\phi}$$

The P-wave velocity is:

$$v_P = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

- **Solenoidal (S-wave) components** (all curl terms):

$$\rho\nabla\times\frac{\partial^2\mathbf{\Psi}}{\partial t^2} = \mu\nabla\times(\nabla^2\mathbf{\Psi})$$

Factor out $\nabla\times$:

$$\nabla\times\left[\rho\frac{\partial^2\mathbf{\Psi}}{\partial t^2} - \mu\nabla^2\mathbf{\Psi}\right] = 0$$

This implies:

$$\boxed{\rho\frac{\partial^2\mathbf{\Psi}}{\partial t^2} = \mu\nabla^2\mathbf{\Psi}}$$

The S-wave velocity is:

$$v_S = \sqrt{\frac{\mu}{\rho}}$$

Physical Interpretation

- **P-waves** (compressional):
  - Involve volume changes ($\nabla\cdot\mathbf{u} \neq 0$)

  - Particle motion parallel to propagation
  - P-waves (compressional) travel faster than S-waves (shear) since $\lambda + 2\mu > \mu$
- **S-waves** (shear):
  - Involve shape changes ($\nabla \times \mathbf{u} \neq 0$)
  - Particle motion perpendicular to propagation
  - Cannot propagate in fluids ($\mu = 0$)
- In homogeneous isotropic media, these wave modes propagate independently.