Elastic Wave From a Point Dislocation Source

OYEKAN, Hammed A.

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Detailed Derivation of the Displacement Field $\mathbf{u}(\mathbf{x}, t)$

This derivation is part of the solution for the elastodynamic Green's function in a homogeneous, unbounded, isotropic medium. For the course **GEOP591 (Theoretical Seismology)**.

We start with the scalar and vector Lamé potentials $\phi(\mathbf{x}, t)$ and $\boldsymbol{\psi}(\mathbf{x}, t)$:

$$\phi(\mathbf{x}, t) = -\frac{1}{4\pi\rho}\left(\frac{\partial}{\partial x_1}\frac{1}{|\mathbf{x}|}\right)\int_0^{|\mathbf{x}|/\alpha} \tau X_0(t-\tau)\,\mathrm{d}\tau, \tag{A}$$

$$\boldsymbol{\psi}(\mathbf{x}, t) = \frac{1}{4\pi\rho}\left(0, \frac{\partial}{\partial x_3}\frac{1}{|\mathbf{x}|}, -\frac{\partial}{\partial x_2}\frac{1}{|\mathbf{x}|}\right)\int_0^{|\mathbf{x}|/\beta} \tau X_0(t-\tau)\,\mathrm{d}\tau. \tag{B}$$

The displacement field is obtained via:

$$\mathbf{u}(\mathbf{x}, t) = \nabla\phi + \nabla\times\boldsymbol{\psi}. \tag{1}$$

1. Detailed Computation of $\nabla\phi$

The gradient of $\phi(\mathbf{x}, t)$ is:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right). \tag{2}$$

From (A), with $r = |\mathbf{x}|$:

$$\phi(\mathbf{x}, t) = -\frac{1}{4\pi\rho}\left(\frac{\partial}{\partial x_1}\frac{1}{r}\right)\int_0^{r/\alpha} \tau X_0(t-\tau)\mathrm{d}\tau. \tag{3}$$

Taking the gradient components:

$$\begin{aligned}\frac{\partial \phi}{\partial x_i} &= -\frac{1}{4\pi\rho}\left(\frac{\partial^2}{\partial x_i \partial x_1}\frac{1}{r}\right)\int_0^{r/\alpha} \tau X_0(t-\tau)\,\mathrm{d}\tau \\ &\quad - \frac{1}{4\pi\rho}\left(\frac{\partial}{\partial x_1}\frac{1}{r}\right)\frac{\partial}{\partial x_i}\left(\int_0^{r/\alpha} \tau X_0(t-\tau)\,\mathrm{d}\tau\right)\end{aligned} \tag{4}$$

Using the Leibniz rule:

$$\frac{\partial}{\partial x_i}\left(\int_0^{r/\alpha} \tau X_0(t-\tau)\mathrm{d}\tau\right) = \left(\frac{r}{\alpha}\right) X_0\left(t-\frac{r}{\alpha}\right)\frac{\partial}{\partial x_i}\left(\frac{r}{\alpha}\right). \tag{5}$$

Since $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$, $\frac{\partial}{\partial x_i}\left(\frac{r}{\alpha}\right) = \frac{x_i}{\alpha r}$:

$$\frac{\partial \phi}{\partial x_i} = -\frac{1}{4\pi\rho}\left[\left(\frac{\partial^2}{\partial x_i \partial x_1}\frac{1}{r}\right)\int_0^{r/\alpha} \tau X_0(t-\tau)\,d\tau + \left(\frac{\partial}{\partial x_1}\frac{1}{r}\right)\frac{\partial}{\partial x_i}\left(\int_0^{r/\alpha} \tau X_0(t-\tau)\,\mathrm{d}\tau\right)\right] \tag{6}$$

$$\frac{\partial \phi}{\partial x_i} = -\frac{1}{4\pi\rho}\left[\left(\frac{\partial^2}{\partial x_i \partial x_1}\frac{1}{r}\right)\int_0^{r/\alpha} \tau X_0(t-\tau)\,d\tau + \left(\frac{\partial}{\partial x_1}\frac{1}{r}\right)\left(\frac{r}{\alpha}\right) X_0\left(t-\frac{r}{\alpha}\right)\frac{\partial}{\partial x_i}\left(\frac{r}{\alpha}\right)\right] \tag{7}$$

$$\boxed{\frac{\partial}{\partial x_1}\frac{1}{r} = -\frac{x_1}{r^3}}$$

Derivation of $\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)$

We want to compute:

$$\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Step 1: Rewrite the expression

$$\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x_1}\left(r^{-1}\right)$$

Step 2: Apply the chain rule

$$\frac{\partial}{\partial x_1}\left(r^{-1}\right) = -r^{-2} \cdot \frac{\partial r}{\partial x_1}$$

Step 3: Compute $\frac{\partial r}{\partial x_1}$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} \implies \frac{\partial r}{\partial x_1} = \frac{x_1}{r}$$

Step 4: Substitute back

$$-r^{-2} \cdot \frac{x_1}{r} = -\frac{x_1}{r^3}$$

Final Result:

$$\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right) = -\frac{x_1}{r^3}$$

$$\frac{\partial \phi}{\partial x_i} = -\frac{1}{4\pi\rho}\left[\left(\frac{\partial^2}{\partial x_i \partial x_1}\frac{1}{r}\right)\int_0^{r/\alpha} \tau X_0(t-\tau)\,d\tau + \left(-\frac{x_1}{r^3}\right)\left(\frac{r}{\alpha}\right) X_0\left(t - \frac{r}{\alpha}\right)\left(\frac{x_i}{\alpha r}\right)\right] \tag{8}$$

$$\boxed{\frac{\partial \phi}{\partial x_i} = -\frac{1}{4\pi\rho}\left(\frac{\partial^2}{\partial x_i \partial x_1}\frac{1}{r}\right)\int_0^{r/\alpha} \tau X_0(t-\tau)\mathrm{d}\tau + \frac{1}{4\pi\rho\alpha^2}\left(\frac{x_i x_1}{r^3}\right) X_0\left(t - \frac{r}{\alpha}\right).}$$

2. Detailed Computation of $\nabla \times \psi$

We compute the curl of the vector potential ψ explicitly using the determinant representation of the curl operator. The vector potential is given by:

$$\psi(x,t) = \frac{1}{4\pi\rho}\left(0, \frac{\partial}{\partial x_3}\frac{1}{r}, -\frac{\partial}{\partial x_2}\frac{1}{r}\right)\int_0^{r/\beta} \tau X_0(t-\tau)\,d\tau$$

where $r = |x|$.

Let's denote:

$$\psi = (0, \psi_2, \psi_3), \quad \text{where} \quad \psi_2 = \frac{1}{4\pi\rho}\left(\frac{\partial}{\partial x_3}\frac{1}{r}\right)F(t), \quad \psi_3 = -\frac{1}{4\pi\rho}\left(\frac{\partial}{\partial x_2}\frac{1}{r}\right)F(t),$$

and

$$F(t) = \int_0^{r/\beta} \tau X_0(t-\tau)\,d\tau$$

Step 1: Determinant Form of Curl

The curl is computed as:

$$\nabla \times \psi = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ 0 & \psi_2 & \psi_3 \end{vmatrix} = \hat{x}_1\left(\frac{\partial \psi_3}{\partial x_2} - \frac{\partial \psi_2}{\partial x_3}\right) - \hat{x}_2\left(\frac{\partial \psi_3}{\partial x_1}\right) + \hat{x}_3\left(\frac{\partial \psi_2}{\partial x_1}\right)$$

Step 2: Component Calculations

x_1**- Component:**

$$(\nabla \times \psi)_1 = \frac{\partial \psi_3}{\partial x_2} - \frac{\partial \psi_2}{\partial x_3}$$

Substitute ψ_2 and ψ_3:

$$(\nabla \times \psi)_1 = \frac{\partial}{\partial x_2}\left(-\frac{1}{4\pi\rho}\frac{\partial}{\partial x_2}\frac{1}{r}F(t)\right) - \frac{\partial}{\partial x_3}\left(\frac{1}{4\pi\rho}\frac{\partial}{\partial x_3}\frac{1}{r}F(t)\right)$$

Apply the product rule:

$$(\nabla\times\psi)_1 = -\frac{1}{4\pi\rho}\left[\frac{\partial^2}{\partial x_2^2}\left(\frac{1}{r}\right)F(t) + \frac{\partial}{\partial x_2}\left(\frac{1}{r}\right)\frac{\partial F(t)}{\partial x_2}\right] - \frac{1}{4\pi\rho}\left[\frac{\partial^2}{\partial x_3^2}\left(\frac{1}{r}\right)F(t) + \frac{\partial}{\partial x_3}\left(\frac{1}{r}\right)\frac{\partial F(t)}{\partial x_3}\right]$$

Simplify using $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{x})$ (Dirac delta function, negligible for $r \neq 0$):

$$\boxed{(\nabla\times\psi)_1 \approx -\frac{1}{4\pi\rho}\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\frac{1}{r}F(t)}$$

x_2-Component:

$$(\nabla\times\psi)_2 = -\frac{\partial\psi_3}{\partial x_1}$$

Substitute ψ_3:

$$\boxed{(\nabla\times\psi)_2 = -\frac{\partial}{\partial x_1}\left(-\frac{1}{4\pi\rho}\frac{\partial}{\partial x_2}\frac{1}{r}F(t)\right) = \frac{1}{4\pi\rho}\frac{\partial^2}{\partial x_1\partial x_2}\left(\frac{1}{r}\right)F(t)}$$

x_3-Component:

$$(\nabla\times\psi)_3 = \frac{\partial\psi_2}{\partial x_1}$$

Substitute ψ_2:

$$\boxed{(\nabla\times\psi)_3 = \frac{\partial}{\partial x_1}\left(\frac{1}{4\pi\rho}\frac{\partial}{\partial x_3}\frac{1}{r}F(t)\right) = \frac{1}{4\pi\rho}\frac{\partial^2}{\partial x_1\partial x_3}\left(\frac{1}{r}\right)F(t)}$$

Step 3: Combine Components and Simplify

The full curl is:

$$\nabla\times\psi = \frac{1}{4\pi\rho}\left[-\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\frac{1}{r}\hat{x}_1 + \frac{\partial^2}{\partial x_1\partial x_2}\frac{1}{r}\hat{x}_2 + \frac{\partial^2}{\partial x_1\partial x_3}\frac{1}{r}\hat{x}_3\right]F(t) \tag{9}$$

Key Simplification:

Recognize that:

$$\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \nabla^2 - \frac{\partial^2}{\partial x_1^2}, \quad \text{and} \quad \nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{x}) \tag{10}$$

For $r \neq 0$, the delta function vanishes, leaving:

$$\nabla \times \psi = \frac{1}{4\pi\rho} \left[\frac{\partial^2}{\partial x_1^2} \frac{1}{r} \hat{x}_1 + \frac{\partial^2}{\partial x_1 \partial x_2} \frac{1}{r} \hat{x}_2 + \frac{\partial^2}{\partial x_1 \partial x_3} \frac{1}{r} \hat{x}_3 \right] F(t) \tag{11}$$

This can be written compactly as:

$$\nabla \times \psi = \frac{1}{4\pi\rho} \left(\frac{\partial^2}{\partial x_1 \partial x_i} \frac{1}{r} \right) F(t) \hat{x}_i \tag{12}$$

Step 4: Include the Time-Dependent Term

The integral $F(t)$ contributes a term when differentiated:

$$\frac{\partial F(t)}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\int_0^{r/\beta} \tau X_0(t-\tau)\, d\tau \right) = \left(\frac{r}{\beta}\right) X_0 \left(t - \frac{r}{\beta}\right) \frac{\partial}{\partial x_i} \left(\frac{r}{\beta}\right) \tag{13}$$

Since $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$, we get:

$$\frac{\partial F(t)}{\partial x_i} = \frac{x_i}{\beta^2} X_0 \left(t - \frac{r}{\beta}\right) \tag{14}$$

Thus, the dominant term in $\nabla \times \psi$ at $r \neq 0$ is:

$$\nabla \times \psi \approx \frac{1}{4\pi\rho\beta^2} \left(\delta_{i1} - \frac{x_i x_1}{r^2}\right) X_0 \left(t - \frac{r}{\beta}\right) \hat{x}_i \tag{15}$$

Final Result for $\nabla \times \psi$

$$\boxed{\nabla \times \psi = \frac{1}{4\pi\rho\beta^2} \left(\delta_{i1} - \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_1}\right) X_0 \left(t - \frac{r}{\beta}\right)}$$

Combine $\nabla\phi$ and $\nabla \times \boldsymbol{\psi}$

The total displacement field is:

$$u_i(x,t) = \frac{\partial \phi}{\partial x_i} + (\nabla \times \psi)$$

Substitute the expressions for $\frac{\partial \phi}{\partial x_i}$ and $(\nabla \times \psi)$:

$$\begin{aligned} u_i(x,t) = &-\frac{1}{4\pi\rho}\left(\frac{\partial^2}{\partial x_i \partial x_1}\frac{1}{r}\right)\int_0^{r/\alpha} \tau X_0(t-\tau)\, d\tau \\ &+ \frac{1}{4\pi\rho\alpha^2}\left(\frac{x_i x_1}{r^3}\right) X_0\left(t - \frac{r}{\alpha}\right) \\ &+ \frac{1}{4\pi\rho\beta^2}\left(\delta_{i1} - \frac{x_i x_1}{r^2}\right) X_0\left(t - \frac{r}{\beta}\right) \end{aligned}$$

The integral term can be split into two parts to highlight the contributions from P-waves (α) and S-waves (β):

$$\int_0^{r/\alpha} \tau X_0(t-\tau)\, d\tau = \int_0^{r/\beta} \tau X_0(t-\tau)\, d\tau - \int_{r/\alpha}^{r/\beta} \tau X_0(t-\tau)\, d\tau$$

Thus, $u_i(x,t)$ becomes:

$$\begin{aligned} u_i(x,t) = &-\frac{1}{4\pi\rho}\frac{\partial^2}{\partial x_i \partial x_1}\left(\frac{1}{r}\right)\int_0^{r/\alpha} \tau X_0(t-\tau)\, d\tau \\ &+ \frac{1}{4\pi\rho\alpha^2}\frac{x_i x_1}{r^3} X_0\left(t - \frac{r}{\alpha}\right) \\ &+ \frac{1}{4\pi\rho}\frac{\partial^2}{\partial x_i \partial x_1}\left(\frac{1}{r}\right)\int_0^{r/\beta} \tau X_0(t-\tau)\, d\tau \\ &+ \frac{1}{4\pi\rho\beta^2}\left(\delta_{i1} - \frac{x_i x_1}{r^2}\right) X_0\left(t - \frac{r}{\beta}\right) \end{aligned}$$

Note on terms:

1. First line: Near-field P-wave contribution (potential term)
2. Second line: Far-field P-wave radiation pattern
3. Third line: Near-field S-wave contribution (curl term)

4. Fourth line: Far-field S-wave radiation pattern

- Corrected integral upper limit from τ/α to r/α for dimensional consistency

Combined Integral Form of Displacement Field

The displacement field can be expressed more compactly by combining the integral terms:

$$\begin{aligned} u_i(x,t) =& \frac{1}{4\pi\rho} \frac{\partial^2}{\partial x_i \partial x_1} \left(\frac{1}{r}\right) \left(\int_0^{r/\beta} - \int_0^{r/\alpha} \right) \tau X_0(t-\tau)\, d\tau \\ &+ \frac{1}{4\pi\rho\alpha^2 r} \left(\frac{x_i x_1}{r^2}\right) X_0\left(t - \frac{r}{\alpha}\right) \\ &+ \frac{1}{4\pi\rho\beta^2 r} \left(\delta_{i1} - \frac{x_i x_1}{r^2}\right) X_0\left(t - \frac{r}{\beta}\right) \end{aligned}$$

The total displacement field is:

$$\boxed{\begin{aligned} u_i(\mathbf{x},t) = & \frac{1}{4\pi\rho} \left(\frac{\partial^2}{\partial x_i \partial x_1} \frac{1}{r} \right) \int_{r/\alpha}^{r/\beta} \tau X_0(t-\tau) \mathrm{d}\tau \\ & + \frac{1}{4\pi\rho\alpha^2 r} \left(\frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_1} \right) X_0\left(t - \frac{r}{\alpha}\right) \\ & + \frac{1}{4\pi\rho\beta^2 r} \left(\delta_{i1} - \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_1} \right) X_0\left(t - \frac{r}{\beta}\right). \end{aligned}}$$

The total displacement can also be written as:

$$\boxed{\begin{aligned} u_i(\mathbf{x},t) =& \frac{1}{4\pi\rho} \left(\frac{\partial^2}{\partial x_i \partial x_1} \frac{1}{r} \right) \int_{r/\alpha}^{r/\beta} \tau X_0(t-\tau)\, d\tau \\ &+ \frac{1}{4\pi\rho\alpha^2 r} \left(\frac{x_i x_1}{r^2}\right) X_0\left(t - \frac{r}{\alpha}\right) \\ &+ \frac{1}{4\pi\rho\beta^2 r} \left(\delta_{i1} - \frac{x_i x_1}{r^2}\right) X_0\left(t - \frac{r}{\beta}\right) \end{aligned}}$$

where:

- α is the P-wave velocity
- β is the S-wave velocity
- $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the radial distance
- $X_0(t)$ is the source time function

- δ_{i1} is the Kronecker delta