

Time Domain Complex-valued ViscoAcoustic Wave Equation [2]

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1 Starting Point: Standard Acoustic Wave Equation

The general inhomogeneous acoustic wave equation in the time domain is:

$$\frac{1}{\rho(x)v^2(x)} \frac{\partial^2 p(x, t)}{\partial t^2} - \nabla \cdot \left[\frac{1}{\rho(x)} \nabla p(x, t) \right] = f(x_s, t) \delta(x - x_s) \quad (1)$$

where $p(x, t)$ is the pressure wavefield, $\rho(x)$ is density, $v(x)$ is velocity, and $f(x_s, t)$ is the source function.

Transform to Frequency Domain

Apply the Fourier transform to equation 1. Throughout this note we adopt the forward/inverse convention

$$P(x, \omega) = \int p(x, t) e^{-i\omega t} dt, \quad p(x, t) = \frac{1}{2\pi} \int P(x, \omega) e^{i\omega t} d\omega, \quad (2)$$

under which $\partial_t \leftrightarrow i\omega$ and hence

$$\mathcal{F} \left\{ \frac{\partial^2 p(x, t)}{\partial t^2} \right\} = -\omega^2 P(x, \omega) \quad (3)$$

This is the convention under which the inverse-transform pairs $i\omega \rightarrow \partial_t$ and $-\omega^2 \rightarrow \partial_{tt}$ used later hold, and for which the Kolsky–Futterman complex velocity below (with $Q > 0$) is causal. All sign choices in the imaginary parts that follow are tied to this convention; adopting $e^{+i\omega t}$ for the forward transform would flip the sign of every imaginary term.

The frequency-domain acoustic wave equation becomes:

$$-\frac{\omega^2}{\rho(x)v^2(x)}P(x,\omega) - \nabla \cdot \left[\frac{1}{\rho(x)} \nabla P(x,\omega) \right] = F(x_s,\omega)\delta(x-x_s) \quad (4)$$

2 Introduce Visco-Acoustic Effects

2.1 Complex-Valued Velocity Model

For visco-acoustic media, the velocity is replaced by a complex-valued, frequency-dependent velocity which describes the viscous attenuation effect in frequency domain (Aki & Richards, 1980) [1].

The complex-valued velocity $v(\mathbf{x},\omega)$ that models both dispersion and dissipation is given by:

$$v(x,\omega) = v_0(x) \left(1 + \underbrace{\frac{1}{\pi Q(x)} \ln \frac{\omega}{\omega_0}}_{\text{Velocity Dispersion}} - \underbrace{i \frac{\text{sgn}(\omega)}{2Q(x)}}_{\text{Amplitude Dissipation}} \right) \quad (5)$$

Strictly, the dispersion term should be read as $\ln(|\omega|/\omega_0)$ so that it is well defined for $\omega < 0$; we keep the customary notation $\ln(\omega/\omega_0)$ with this understanding.

The quantities in this expression are:

- $v_0(\mathbf{x})$: The reference velocity at a reference frequency ω_0 .
- $\frac{1}{\pi Q(x)} \ln \frac{\omega}{\omega_0}$: The term responsible for **velocity dispersion** (frequency-dependent phase speed).
- $Q(x)$ is the quality factor (frequency-independent)
- $-i \frac{\text{sgn}(\omega)}{2Q(x)}$: The term responsible for **amplitude dissipation** (frequency-dependent energy loss). It characterizes amplitude loss.
- **sgn** is the sign (signum) function: $\text{sgn}(x) = (1 \text{ if } x > 0, 0 \text{ if } x = 0, -1 \text{ if } x < 0)$

2.2 Frequency-Domain Visco-Acoustic Wave Equation

Substituting the complex velocity (Eq. 5) into the frequency-domain wave equation (Eq. 4):

$$-\frac{\omega^2}{\rho(x)v_0^2(x) \left(1 + \frac{1}{\pi Q(x)} \ln \frac{\omega}{\omega_0} - i \frac{\text{sgn}(\omega)}{2Q(x)} \right)^2} P(x,\omega) - \nabla \cdot \left[\frac{1}{\rho(x)} \nabla P(x,\omega) \right] = F(x_s,\omega)\delta(x-x_s) \quad (6)$$

For weak attenuation (high Q , $1/Q$ is small for weak attenuation), we can expand the denominator and neglect higher-order terms in $1/Q$ (using first-order expansion). This leads to:

$$\left(1 + \frac{1}{\pi Q} \ln \frac{\omega}{\omega_0} - i \frac{\text{sgn}(\omega)}{2Q}\right)^{-2} \approx 1 - \frac{2}{\pi Q} \ln \frac{\omega}{\omega_0} + i \frac{\text{sgn}(\omega)}{Q} \quad (7)$$

Equation 6 then becomes:

$$-\frac{\omega^2}{\rho(x)v_0^2(x)} \left[1 - \frac{2}{\pi Q(x)} \left(\ln \frac{\omega}{\omega_0} + i \frac{\text{sgn}(\omega)}{Q(x)} \right) \right] P(x, \omega) - \nabla \cdot \left[\frac{1}{\rho(x)} \nabla P(x, \omega) \right] = F(x_s, \omega) \delta(x - x_s) \quad (8)$$

The terms $\omega^2 \ln \frac{\omega}{\omega_0}$ and $i\omega^2 \text{sgn}(\omega)$ are still difficult to transform directly back to the time domain. So, we need to approximate these two terms as well.

3 Approximations for Time-Domain Transformation

The challenge is that the **logarithmic** and **the signum functions** make direct inverse Fourier transformation difficult. Two approximations are needed.

3.1 Dispersion Approximation

The authors propose approximating the logarithmic dispersion term with a second-order polynomial in ω

$$\omega^2 \ln \frac{\omega}{\omega_0} \approx a\omega^2 + b\omega + c \quad (9)$$

where a , b , and c are coefficients determined by least-squares fitting over the frequency band of interest (e.g., 1-150 Hz).

Substituting this approximation into the dispersion term in Equation 8:

$$\begin{aligned} -\frac{\omega^2}{\rho(x)v_0^2(x)} \left[1 - \frac{2}{\pi Q(x)} \ln \frac{\omega}{\omega_0} \right] P(x, \omega) \\ \approx -\frac{1}{\rho(x)v_0^2(x)} \left[\omega^2 - \frac{2}{\pi Q(x)} (a\omega^2 + b\omega + c) \right] P(x, \omega) \end{aligned} \quad (10)$$

Rearranging:

$$\boxed{-\frac{1}{\rho(x)v_0^2(x)} \left[\left(1 - \frac{2a}{\pi Q(x)}\right) \omega^2 - \frac{2b}{\pi Q(x)} \omega - \frac{2c}{\pi Q(x)} \right] P(x, \omega)} \quad (11)$$

This is equation (5) from the paper and it represents the approximation for the **dispersion term**.

This is a crucial step. The frequency-domain operator is now a simple polynomial in ω , which is easy to transform to the time domain using the properties of the Fourier transform.

3.2 Dissipation Approximation

The dissipation contribution in Eq. (8) is the term

$$-i \frac{\omega^2 \operatorname{sgn}(\omega)}{\rho v_0^2 Q} P(x, \omega).$$

For the manipulations that follow we work with the factor

$$i \frac{\omega^2 \operatorname{sgn}(\omega)}{\rho v_0^2 Q},$$

and restore the overall minus sign in the boxed result (Eq. (14)).

The presence of $\operatorname{sgn}(\omega)$ makes direct transformation difficult. The authors use an approximation from Zhu & Harris (2014) [3], which is valid for weak attenuation. The key insight is to relate the frequency ω to the wavenumber k .

$$k \approx k_r \quad \text{and} \quad k_r \approx \frac{\omega}{v_0} \quad (12)$$

where k is the complex wavenumber and k_r is its real part.

For a propagating wave, $\omega \approx v_0 |k|$. Therefore:

$$|\omega| \approx v_0 |k| \approx k_r v_0$$

Using this, they approximate the term as follows:

$$\begin{aligned} \frac{i\omega^2 \operatorname{sgn}(\omega)}{\rho v_0^2 Q} &= \frac{i\omega^2}{\rho v_0^2 Q} \cdot \frac{|\omega|}{\omega} \approx \frac{i\omega^2}{\rho v_0^2 Q} \cdot \frac{v_0 |k|}{\omega} \\ &= \frac{i\omega |k|}{\rho v_0 Q} \end{aligned} \quad (13)$$

In the space domain:

The wavenumber k corresponds to the spatial derivative operator $-i\nabla$, so $|k|$ corresponds to the square root of the negative Laplacian, $\sqrt{-\nabla^2}$. This gives the final approximation for the dissipation term:

$$\boxed{-i \frac{\omega^2 \text{sgn}(\omega)}{\rho(x)v_0^2(x)Q(x)} P(x, \omega) \approx -i \frac{\omega |k|}{\rho(x)v_0(x)Q(x)} P(x, \omega) \approx -i \frac{\omega \sqrt{-\nabla^2}}{\rho(x)v_0(x)Q(x)} P(x, \omega)} \quad (14)$$

This is equation (8) from the paper.

Where:

- $|\cdot|$ denotes the absolute value.
- $\sqrt{-\nabla^2}$ is known as the **fractional Laplacian**, a pseudo-differential operator

4 Assembling the Approximated Frequency-Domain Equation

Now, we plug the two approximations (Eq. (11) and Eq. (14)) back into the original equation (Eq. (8)).

Original Eq. (8):

$$-\frac{\omega^2}{\rho v_0^2} \left[1 - \frac{2}{\pi Q} \ln \frac{\omega}{\omega_0} + i \frac{\text{sgn}(\omega)}{Q} \right] P - \nabla \cdot \left[\frac{1}{\rho} \nabla P \right] = F \delta$$

Substitute with Approximations:

1. Substitute the dispersion part from **Eq. (11)**.
2. Substitute the dissipation part from **Eq. (14)**.

This yields the approximated frequency-domain visco-acoustic equation:

$$\boxed{-\frac{1}{\rho v_0^2} \left[\left(1 - \frac{2a}{\pi Q} \right) \omega^2 - \frac{2b}{\pi Q} \omega - \frac{2c}{\pi Q} \right] P - i \frac{\omega \sqrt{-\nabla^2}}{\rho v_0 Q} P - \nabla \cdot \left(\frac{1}{\rho} \nabla P \right) = F \delta} \quad (15)$$

where the coefficients a, b, c, ρ, v_0 and Q carry their spatial dependence ($\mathbf{x}$) as before.

5 Inverse Fourier Transform to the Time Domain

Finally, we apply the **inverse Fourier transform** to the entire equation (15).

Fourier Transform Pairs

- $i\omega \rightarrow \frac{\partial}{\partial t}$ (First time derivative)
- $-\omega^2 \rightarrow \frac{\partial^2}{\partial t^2}$ (Second time derivative)
- $P(\mathbf{x}, \omega) \rightarrow p(\mathbf{x}, t)$ (Wavefield)
- $F(\mathbf{x}_s, \omega) \rightarrow f(\mathbf{x}_s, t)$ (Source)
- The spatial operators ∇ , $\nabla \cdot$, and $\sqrt{-\nabla^2}$ remain unchanged.

Let's transform Eq. (15) term-by-term.

1. First Term (Dispersion Approximation):

$$-\frac{1}{\rho v_0^2} \left[\left(1 - \frac{2a}{\pi Q} \right) \omega^2 P - \frac{2b}{\pi Q} \omega P - \frac{2c}{\pi Q} P \right]$$

$$\xrightarrow{\mathcal{F}^{-1}} -\frac{1}{\rho v_0^2} \left[\left(1 - \frac{2a}{\pi Q} \right) \frac{\partial^2 p}{\partial t^2} - i \frac{2b}{\pi Q} \frac{\partial p}{\partial t} + \frac{2c}{\pi Q} p \right]$$

Note: The sign changes because $-\omega^2 \rightarrow \frac{\partial^2}{\partial t^2}$ and $i\omega \rightarrow \frac{\partial}{\partial t}$.

2. Second Term (Dissipation Approximation):

$$-\frac{i\omega\sqrt{-\nabla^2}}{\rho v_0 Q} P \xrightarrow{\mathcal{F}^{-1}} -\frac{\sqrt{-\nabla^2}}{\rho v_0 Q} \frac{\partial p}{\partial t}$$

Note: $i\omega P \rightarrow \frac{\partial p}{\partial t}$.

3. Third Term (Laplacian):

$$-\nabla \cdot \left[\frac{1}{\rho} \nabla P \right] \xrightarrow{\mathcal{F}^{-1}} -\nabla \cdot \left[\frac{1}{\rho} \nabla p \right]$$

4. Source Term:

$$F(x_s, \omega) \delta(x - x_s) \xrightarrow{\mathcal{F}^{-1}} f(x_s, t) \delta(x - x_s)$$

Now, let us assemble all these transformed terms into the final time-domain equation. Let's also define the coefficients C_1, C_2, C_3, C_4 for clarity.

The final Time-Domain Complex-Valued Visco-Acoustic Wave Equation is:

$$\boxed{\underbrace{\frac{1}{\rho(\mathbf{x})v_0^2(\mathbf{x})} \left[C_1 \frac{\partial^2 p}{\partial t^2} - iC_2 \frac{\partial p}{\partial t} + C_3 p \right]}_{\text{Dispersion-dominated}} - \underbrace{C_4 \sqrt{-\nabla^2} \frac{\partial p}{\partial t}}_{\text{Dissipation-dominated}} - \nabla \cdot \left[\frac{1}{\rho(\mathbf{x})} \nabla p \right] = f(\mathbf{x}_s, t) \delta(\mathbf{x} - \mathbf{x}_s)} \quad (16)$$

Where the coefficients are:

$$\begin{aligned} C_1 &= \left(1 - \frac{2a}{\pi Q(\mathbf{x})} \right), & C_2 &= \frac{2b}{\pi Q(\mathbf{x})}, \\ C_3 &= \frac{2c}{\pi Q(\mathbf{x})}, & C_4 &= \frac{1}{\rho(\mathbf{x})v_0(\mathbf{x})Q(\mathbf{x})} \end{aligned} \quad (17)$$

Note on the imaginary sign of C_2 :

Note that in equation (16), we have $-i \frac{2b}{\pi Q}$. The presence of the imaginary unit i in the C_2 term is what makes the wavefield solution complex-valued.

6 Features of the Derived Equation

1. **Complex-valued:** The imaginary unit i in the C_2 term makes the wavefield complex-valued, coupling the real and imaginary parts during propagation.
2. **Dispersion:** The three terms with coefficients C_1, C_2, C_3 represent the polynomial approximation to the logarithmic dispersion.
3. **Dissipation:** The fractional Laplacian term $C_4 \sqrt{-\nabla^2} \frac{\partial p(x,t)}{\partial t}$ represents amplitude attenuation.
4. **Flexibility:** By setting different coefficients to zero, we can obtain:
 - Dispersion-dominated equation: Set $C_4 = 0$
 - Dissipation-dominated equation: Set $C_1 = 1, C_2 = C_3 = 0$

7 Physical Interpretation

- The standard wave equation only has the second-derivative term $\frac{\partial^2 p}{\partial t^2}$

- Visco-acoustic effects add:
 - Modified second derivative ($C_1 \neq 1$): Velocity dispersion
 - First derivative with i ($-iC_2 \frac{\partial p}{\partial t}$): Phase coupling due to dispersion
 - Zeroth-order term ($C_3 p$): Frequency-dependent baseline
 - Fractional derivative ($C_4 \sqrt{-\nabla^2} \frac{\partial p}{\partial t}$): Non-local dissipation

8 Discretization of the Time-Domain Complex-Valued Visco-Acoustic Wave Equation

The general time-domain visco-acoustic wave equation is written as

$$\frac{1}{\rho(\mathbf{x})v_0^2(\mathbf{x})} \left[C_1 \frac{\partial^2 p(\mathbf{x}, t)}{\partial t^2} - iC_2 \frac{\partial p(\mathbf{x}, t)}{\partial t} + C_3 p(\mathbf{x}, t) \right] - C_4 \sqrt{-\nabla^2} \frac{\partial p(\mathbf{x}, t)}{\partial t} - \nabla \cdot \left[\frac{1}{\rho(\mathbf{x})} \nabla p(\mathbf{x}, t) \right] = f(\mathbf{x}_s, t) \delta(\mathbf{x} - \mathbf{x}_s). \quad (18)$$

We introduce the shorthand

$$m(\mathbf{x}) = \frac{1}{\rho(\mathbf{x})v_0^2(\mathbf{x})}, \quad \mathcal{S} = \sqrt{-\nabla^2}, \quad s(\mathbf{x}, t) = f(\mathbf{x}_s, t) \delta(\mathbf{x} - \mathbf{x}_s),$$

so that Eq. (18) can be written compactly as

$$m(\mathbf{x}) \left[C_1 \partial_{tt} p - iC_2 \partial_t p + C_3 p \right] - C_4 \mathcal{S} \partial_t p - \nabla \cdot \left[\frac{1}{\rho} \nabla p \right] = s. \quad (19)$$

Time discretization

Let $p^n(\mathbf{x}) = p(\mathbf{x}, t_n)$ where $t_n = n\Delta t$. We use centered finite differences for the time derivatives:

$$\partial_{tt} p|_{t_n} \approx \frac{p^{n+1} - 2p^n + p^{n-1}}{\Delta t^2}, \quad (20)$$

$$\partial_t p|_{t_n} \approx \frac{p^{n+1} - p^{n-1}}{2\Delta t}. \quad (21)$$

The fractional operator acting on the time derivative is also centered:

$$\mathcal{S} \partial_t p|_{t_n} \approx \mathcal{S} \left(\frac{p^{n+1} - p^{n-1}}{2\Delta t} \right). \quad (22)$$

The spatial Laplacian term is evaluated explicitly at time n :

$$\text{Div}^n = \nabla \cdot \left(\frac{1}{\rho} \nabla p^n \right). \quad (23)$$

With pseudospectral method $\mathcal{S}$ and Div are computed via FFTs: $\mathcal{S} \leftrightarrow |\mathbf{k}|$, $-\nabla^2 \leftrightarrow |\mathbf{k}|^2$, etc.

Fully discrete equation

Substituting these approximations into Eq. (19), we obtain

$$m \left[C_1 \frac{p^{n+1} - 2p^n + p^{n-1}}{\Delta t^2} - iC_2 \frac{p^{n+1} - p^{n-1}}{2\Delta t} + C_3 p^n \right] - C_4 \mathcal{S} \frac{p^{n+1} - p^{n-1}}{2\Delta t} - \text{Div}^n = s^n. \quad (24)$$

We collect terms multiplying p^{n+1} , p^n , and p^{n-1} :

$$A(\mathbf{x}) = m \left(\frac{C_1}{\Delta t^2} - \frac{iC_2}{2\Delta t} \right) - \frac{C_4}{2\Delta t} \mathcal{S}, \quad (25)$$

$$B(\mathbf{x}) = m \left(-\frac{2C_1}{\Delta t^2} + C_3 \right), \quad (26)$$

$$C(\mathbf{x}) = m \left(\frac{C_1}{\Delta t^2} + \frac{iC_2}{2\Delta t} \right) + \frac{C_4}{2\Delta t} \mathcal{S}. \quad (27)$$

Equation (24) becomes

$$A p^{n+1} + B p^n + C p^{n-1} - \text{Div}^n = s^n. \quad (28)$$

Rearranging for p^{n+1} :

$$A p^{n+1} = s^n + \text{Div}^n - B p^n - C p^{n-1}. \quad (29)$$

To emphasize positive coefficients, define

$$D = m \left(\frac{2C_1}{\Delta t^2} - C_3 \right), \quad E = m \left(\frac{C_1}{\Delta t^2} + \frac{iC_2}{2\Delta t} \right) + \frac{C_4}{2\Delta t} \mathcal{S},$$

so that the discrete update equation becomes

$$A p^{n+1} = s^n + \text{Div}^n + D p^n - E p^{n-1}. \quad (30)$$

Explicit update for p^{n+1}

Finally, the update formula for the new wavefield p^{n+1} is

$$\boxed{p^{n+1} = A^{-1} (s^n + \text{Div}^n + D p^n - E p^{n-1})}. \quad (31)$$

Because $A = m\left(\frac{C_1}{\Delta t^2} - \frac{iC_2}{2\Delta t}\right) - \frac{C_4}{2\Delta t}\mathcal{S}$ contains the nonlocal operator $\mathcal{S} = \sqrt{-\nabla^2}$, the notation A^{-1} denotes an *operator* inverse rather than a pointwise division. For constant ρ , v_0 and Q (homogeneous medium), A is diagonal in the wavenumber domain, $\hat{A}(\mathbf{k}) = m\left(\frac{C_1}{\Delta t^2} - \frac{iC_2}{2\Delta t}\right) - \frac{C_4}{2\Delta t}|\mathbf{k}|$, and the inverse is applied exactly and pointwise in $\mathbf{k}$ -space via FFT. For heterogeneous media A is not diagonalised by a single FFT; one then either solves the implicit system each step, or — more commonly — treats the $C_4\mathcal{S}\partial_t p$ term explicitly (evaluating $\mathcal{S}\partial_t p$ at time level n), which removes $\mathcal{S}$ from A and makes the update fully explicit and pointwise. This explicit treatment is exactly the decoupled scheme of Yang & Zhu (2018): they split Eq. (16) into a complex-valued FDTD equation plus an auxiliary wavefield

$$q(\mathbf{x}, t) = \sqrt{-\nabla^2} \frac{\partial p(\mathbf{x}, t)}{\partial t} = \mathcal{F}^{-1} \left\{ |\mathbf{k}| \mathcal{F} \left[\frac{\partial p}{\partial t} \right] \right\},$$

evaluated pseudospectrally from known time levels, so that $C_4 q$ enters the update explicitly and no operator inverse is required.

The coefficients are summarized as

$$A = m\left(\frac{C_1}{\Delta t^2} - \frac{iC_2}{2\Delta t}\right) - \frac{C_4}{2\Delta t}\mathcal{S}, \quad (32)$$

$$D = m\left(\frac{2C_1}{\Delta t^2} - C_3\right), \quad (33)$$

$$E = m\left(\frac{C_1}{\Delta t^2} + \frac{iC_2}{2\Delta t}\right) + \frac{C_4}{2\Delta t}\mathcal{S}. \quad (34)$$

Here all coefficients are generally complex-valued, and both the real and imaginary parts of p evolve simultaneously during propagation.

Implementation notes

In a pseudospectral framework:

- The operator $\mathcal{S} = \sqrt{-\nabla^2}$ is applied in the wavenumber domain:

$$\widehat{\mathcal{S}p}(\mathbf{k}) = |\mathbf{k}| \widehat{p}(\mathbf{k}).$$

- The Laplacian term is computed as

$$\text{Div}^n = \nabla \cdot \left(\frac{1}{\rho} \nabla p^n \right),$$

using FFT-based gradients and divergence operators.

- p must be treated as a complex-valued field since coefficients C_2 and C_4 introduce complex coupling between real and imaginary parts.

Equation (31) thus provides a fully discrete time-stepping scheme for the complex-valued visco-acoustic wave equation.

References

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- [3] Tieyuan Zhu and Jerry M. Harris. “Modeling acoustic wave propagation in heterogeneous attenuating media using decoupled fractional Laplacians”. In: *Geophysics* 79.3 (2014), T105–T116.