Analytical Solution to 2D SH (Shear Horizontal) ViscoElastic Wave Equation Using Cagniard–De Hoop Method

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Introduction

To describe a viscoelastic medium, we modify the stress–strain relation because the conservation of momentum is independent of the material behaviour. The viscoelastic stress–strain relation generalises the purely elastic case by introducing frequency-dependent complex moduli (or quality factor, Q), or, equivalently, time-domain convolution integrals described by the Boltzmann superposition and causality principles:

$$\sigma(t) = \int_{-\infty}^{t} \Psi(t - t')\, \dot{\epsilon}(t')\, dt'$$

where $\Psi(t)$ is the relaxation function.

1 Governing Wave Equation

2D SH Wave:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} + f_y \tag{1}$$

$$\sigma_{yx} = \mu \frac{\partial u_y}{\partial x} \tag{2}$$

$$\sigma_{yz} = \mu \frac{\partial u_y}{\partial z} \tag{3}$$

1D SH Wave:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \sigma_{yx}}{\partial x} + f_y \tag{4}$$

- ρ: mass density (kg/m^3)
- $u_y(x, z, t)$: transverse displacement field (m)
- σ_{yx}, σ_{yz}: shear stress components (Pa)
- f_y: external force density (N/m^3)
- x, z: spatial coordinates
- t: time coordinate

2 ViscoElastic Wave Equation

Kelvin–Voigt Model:

The Kelvin–Voigt model adds a velocity-dependent (strain-rate) damping term to the stress–strain relation. The modified stress components become:

$$\sigma_{yx} = \mu \frac{\partial u_y}{\partial x} + \zeta \frac{\partial}{\partial t}\left(\frac{\partial u_y}{\partial x}\right) \tag{5}$$

$$\sigma_{yz} = \mu \frac{\partial u_y}{\partial z} + \zeta \frac{\partial}{\partial t}\left(\frac{\partial u_y}{\partial z}\right) \tag{6}$$

Where ζ is the viscosity coefficient. Substituting these into the equation of motion:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + \zeta \frac{\partial}{\partial t}\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + A\delta(x)\delta(z)\delta(t) \tag{7}$$

Let $\beta^2 = \mu/\rho$ (elastic shear-wave speed squared) and introduce the viscous parameter $\gamma = \zeta/\rho$ called the kinematic viscosity.

Dividing (7) by ρ, the equation becomes:

$$\boxed{\frac{\partial^2 u_y}{\partial t^2} = \beta^2 \nabla^2 u_y + \gamma \frac{\partial}{\partial t}\nabla^2 u_y + \frac{A}{\rho}\delta(x)\delta(z)\delta(t)} \tag{8}$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

We seek to find the causal Green's function $u_y(x, z, t)$ for the wave equation in (8), i.e., the response to the source $A\delta(x)\delta(z)\delta(t)$.

3 The Cagniard–De Hoop Method

Given the 2D ViscoElastic SH wave equation using the Kelvin–Voigt Model:

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + \zeta \frac{\partial}{\partial t}\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + A\delta(x)\delta(z)\delta(t)$$

3.1 Step 1: Take the Laplace Transform with Respect to Time

The Laplace transform of a function $f(t)$ is defined as:

$$\mathcal{L}\{f(t)\} = \tilde{f}(s) = \int_0^\infty f(t)e^{-st}dt$$

Apply the Laplace transform to both sides of the equation. We will use the following properties of the Laplace transform:

1. $\mathcal{L}\left\{\frac{\partial^2 u_y}{\partial t^2}\right\} = s^2\tilde{u}_y - su_y(x,z,0) - \frac{\partial u_y}{\partial t}(x,z,0)$
2. $\mathcal{L}\left\{\frac{\partial}{\partial t}\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right)\right\} = s\left(\frac{\partial^2 \tilde{u}_y}{\partial x^2} + \frac{\partial^2 \tilde{u}_y}{\partial z^2}\right) - \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right)\Bigg|_{t=0}$
3. $\mathcal{L}\{\delta(t)\} = 1$

Assuming initial conditions are zero (quiescent IC):

$$u_y(x,z,0) = 0, \quad \frac{\partial u_y}{\partial t}(x,z,0) = 0, \quad \text{and} \quad \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right)\Bigg|_{t=0} = 0$$

The Laplace transform simplifies the equation to:

$$\rho\left(s^2\tilde{u}_y\right) = \mu\left(\frac{\partial^2 \tilde{u}_y}{\partial x^2} + \frac{\partial^2 \tilde{u}_y}{\partial z^2}\right) + \zeta s\left(\frac{\partial^2 \tilde{u}_y}{\partial x^2} + \frac{\partial^2 \tilde{u}_y}{\partial z^2}\right) + A\delta(x)\delta(z)$$

Combine Like Terms

Combine the terms involving $\frac{\partial^2 \tilde{u}_y}{\partial x^2} + \frac{\partial^2 \tilde{u}_y}{\partial z^2}$:

$$\rho s^2\tilde{u}_y = (\mu + \zeta s)\left(\frac{\partial^2 \tilde{u}_y}{\partial x^2} + \frac{\partial^2 \tilde{u}_y}{\partial z^2}\right) + A\delta(x)\delta(z)$$

Rearrange the Equation

Divide both sides by ρ to isolate $\tilde{u}_y$:

$$s^2\tilde{u}_y(x,z,s) = \frac{\mu + \zeta s}{\rho}\left(\frac{\partial^2 \tilde{u}_y}{\partial x^2} + \frac{\partial^2 \tilde{u}_y}{\partial z^2}\right) + \frac{A}{\rho}\delta(x)\delta(z)$$

Let $\beta^2 = \frac{\mu}{\rho}$ and $\gamma = \frac{\zeta}{\rho}$, then the equation becomes:

$$s^2\tilde{u}_y(x,z,s) = (\beta^2 + \gamma s)\left(\frac{\partial^2 \tilde{u}_y}{\partial x^2} + \frac{\partial^2 \tilde{u}_y}{\partial z^2}\right) + \frac{A}{\rho}\delta(x)\delta(z)$$

Final Form of the Laplace-Transformed Equation

$$\boxed{\frac{\partial^2 \tilde{u}_y}{\partial x^2} + \frac{\partial^2 \tilde{u}_y}{\partial z^2} - \frac{s^2}{\beta^2 + \gamma s}\tilde{u}_y = -\frac{A}{\rho(\beta^2 + \gamma s)}\delta(x)\delta(z)} \tag{9}$$

Physical reading. Define the complex, s-dependent "viscoelastic" shear speed $c^2(s) = \beta^2 + \gamma s$. The medium behaves like an elastic medium whose wavespeed depends on the Laplace variable. This single fact is the source of every difficulty in the Cagniard step below.

3.2 Step 2: Take the Fourier Transform in the x-Direction

The Fourier transform of a function $f(x)$ is defined as:

$$\mathcal{F}\{f(x)\} = \hat{f}(k_x) = \int_{-\infty}^{\infty} f(x)e^{-ik_x x}dx$$

Apply the Fourier transform to both sides of the equation. We will use the following properties of the Fourier transform:

1. $\mathcal{F}\left\{\frac{\partial^2 \tilde{u}_y}{\partial x^2}\right\} = -k_x^2 \hat{\tilde{u}}_y$
2. $\mathcal{F}\left\{\frac{\partial^2 \tilde{u}_y}{\partial z^2}\right\} = \frac{\partial^2 \hat{\tilde{u}}_y}{\partial z^2}$ (since the FT is taken only in x)
3. $\mathcal{F}\{\tilde{u}_y\} = \hat{\tilde{u}}_y$
4. $\mathcal{F}\{\delta(x)\} = 1$

Applying the Fourier transform to the equation (9):

$$\mathcal{F}\left\{\frac{\partial^2 \tilde{u}_y}{\partial x^2}\right\} + \mathcal{F}\left\{\frac{\partial^2 \tilde{u}_y}{\partial z^2}\right\} - \frac{s^2}{\beta^2 + \gamma s}\mathcal{F}\{\tilde{u}_y\} = -\frac{A}{\rho(\beta^2 + \gamma s)}\mathcal{F}\{\delta(x)\}\delta(z)$$

Substituting the FT properties:

$$-k_x^2 \hat{\tilde{u}}_y + \frac{\partial^2 \hat{\tilde{u}}_y}{\partial z^2} - \frac{s^2}{\beta^2 + \gamma s}\hat{\tilde{u}}_y = -\frac{A}{\rho(\beta^2 + \gamma s)} \cdot 1 \cdot \delta(z)$$

Combine the terms involving $\hat{\tilde{u}}_y$:

$$\frac{\partial^2 \hat{\tilde{u}}_y}{\partial z^2} - \left(k_x^2 + \frac{s^2}{\beta^2 + \gamma s}\right)\hat{\tilde{u}}_y = -\frac{A}{\rho(\beta^2 + \gamma s)}\delta(z)$$

Let $\kappa^2 = k_x^2 + \frac{s^2}{\beta^2+\gamma s}$, then the equation simplifies to:

$$\boxed{\frac{\partial^2 \hat{\tilde{u}}_y}{\partial z^2} - \kappa^2(k_x, s)\hat{\tilde{u}}_y(k_x, z, s) = -\frac{A}{\rho(\beta^2 + \gamma s)}\delta(z)} \tag{10}$$

$$\boxed{\kappa^2(k_x, s) = k_x^2 + \frac{s^2}{\beta^2 + \gamma s}}$$

3.3 Step 3: Solve the ODE in the z-Direction

The equation (10) above is a second-order ordinary differential equation (ODE) in z with a delta function source.

The general solution for $z \neq 0$ is:

3.3.1 Homogeneous Equation Solution

The homogeneous equation is:

$$\frac{d^2 \hat{\tilde{u}}_y}{dz^2} - \kappa^2 \hat{\tilde{u}}_y = 0$$

This is a linear second-order ODE with constant coefficients. The general solution is:

$$\hat{\tilde{u}}_y^{\text{hom}}(z) = \begin{cases} C_1 e^{\kappa z} + C_2 e^{-\kappa z} & z < 0 \\ C_3 e^{\kappa z} + C_4 e^{-\kappa z} & z > 0 \end{cases}$$

But we require **boundedness at infinity**:

- As $z \to -\infty$, $e^{-\kappa z} \to \infty$ (since $\mathrm{Re}(\kappa) > 0$) $\implies C_2 = 0$
- As $z \to +\infty$, $e^{\kappa z} \to \infty$ (since $\mathrm{Re}(\kappa) > 0$) $\implies C_3 = 0$

So the physically admissible (bounded) solution becomes:

$$\hat{\tilde{u}}_y^{\text{hom}}(z) = \begin{cases} C_1 e^{\kappa z} & z < 0 \\ C_4 e^{-\kappa z} & z > 0 \end{cases}$$

3.3.2 Apply Discontinuity (Jump) Condition from δ-function

To account for the delta function at $z = 0$, we impose continuity and a jump condition in the derivative:

1. **Continuity at $z = 0$:**

$$\hat{\tilde{u}}_y(0^+) = \hat{\tilde{u}}_y(0^-)$$

2. **Jump condition in the derivative:** Integrate the ODE across $z = 0$:

$$\int_{0^-}^{0^+} \frac{\partial^2 \hat{\tilde{u}}_y}{\partial z^2} dz - \kappa^2 \int_{0^-}^{0^+} \hat{\tilde{u}}_y dz = -\frac{A}{\rho(\beta^2 + \gamma s)} \int_{0^-}^{0^+} \delta(z) dz$$

The first integral gives the jump in the derivative:

$$\left. \frac{\partial \hat{\tilde{u}}_y}{\partial z} \right|_{0^+} - \left. \frac{\partial \hat{\tilde{u}}_y}{\partial z} \right|_{0^-} = -\frac{A}{\rho(\beta^2 + \gamma s)}$$

The second integral vanishes because $\hat{\tilde{u}}_y$ is continuous.

For a symmetric solution (assuming decay as $|z| \to \infty$):

- For $z > 0$: $\hat{\tilde{u}}_y(z) = Ce^{-\kappa z}$
- For $z < 0$: $\hat{\tilde{u}}_y(z) = Ce^{\kappa z}$

Applying the jump condition:

$$\left. \frac{\partial \hat{\tilde{u}}_y}{\partial z} \right|_{0^+} = -\kappa C, \quad \left. \frac{\partial \hat{\tilde{u}}_y}{\partial z} \right|_{0^-} = \kappa C$$

Thus:

$$-\kappa C - \kappa C = -\frac{A}{\rho(\beta^2 + \gamma s)} \implies -2\kappa C = -\frac{A}{\rho(\beta^2 + \gamma s)}$$

Solving for C:

$$C = \frac{A}{2\kappa\rho(\beta^2 + \gamma s)}$$

Hence the Fourier–Laplace solution is

$$\boxed{\hat{\tilde{u}}_y(k_x, z, s) = \frac{A}{2\rho(\beta^2 + \gamma s)} \frac{e^{-\kappa|z|}}{\kappa}, \qquad \kappa = \sqrt{k_x^2 + \frac{s^2}{\beta^2 + \gamma s}}} \tag{11}$$

This is the **Fourier–Laplace transformed displacement field** $\hat{\tilde{u}}_y(k_x, z, s)$ for the viscoelastic wave equation under a point force source.

3.4 Step 4: Take the Inverse Spatial Fourier Transform of the Fourier–Laplace Transformed Solution

Given Fourier-Laplace Space Solution

$$\hat{\tilde{u}}_y(k_x, z, s) = \frac{A}{2\kappa\rho(\beta^2 + \gamma s)} e^{-\kappa|z|}$$

where

$$\kappa = \sqrt{k_x^2 + \frac{s^2}{\beta^2 + \gamma s}}$$

$$\hat{\tilde{u}}_y(k_x, z, s) = \frac{A}{2\rho(\beta^2 + \gamma s)} \cdot \frac{e^{-\kappa|z|}}{\kappa}$$

Inverse Fourier Transform in x

The inverse Fourier transform is defined as:

$$\tilde{u}_y(x, z, s) = \mathcal{F}^{-1}\{\hat{\tilde{u}}_y(k_x, z, s)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\tilde{u}}_y(k_x, z, s) e^{ik_x x} \, dk_x$$

Substituting $\hat{\tilde{u}}_y$:

$$\tilde{u}_y(x, z, s) = \frac{A}{4\pi\rho(\beta^2 + \gamma s)} \int_{-\infty}^{\infty} \frac{e^{-\kappa|z|}}{\kappa} e^{ik_x x} \, dk_x$$

Simplify the Integral

The integral to solve is:

$$I = \int_{-\infty}^{\infty} \frac{e^{-\kappa|z|}}{\kappa} e^{ik_x x} \, dk_x$$

Let $\kappa = \sqrt{k_x^2 + \kappa_s^2}$, where $\kappa_s^2 = \frac{s^2}{\beta^2+\gamma s}$. The integral becomes:

$$I = \int_{-\infty}^{\infty} \frac{e^{-\sqrt{k_x^2+\kappa_s^2}\,|z|}}{\sqrt{k_x^2 + \kappa_s^2}} e^{ik_x x} \, dk_x$$

Using Known Integral Results

The integral above is a known integral representation of the modified Bessel function of the second kind K_0. This integral is a standard form (Gradshteyn–Ryzhik; the 2D free-space Helmholtz/Green identity) whose result is known from tables of Fourier transforms or Green's functions [1]. The result is:

$$\int_{-\infty}^{\infty} \frac{e^{-\sqrt{k_x^2+\kappa_s^2}\,|z|}}{\sqrt{k_x^2 + \kappa_s^2}} e^{ik_x x} dk_x = 2K_0(\kappa_s \sqrt{x^2 + z^2})$$

where K_0 is the modified Bessel function of the second kind of order zero, valid for $\mathrm{Re}(\kappa_s) > 0$.

Another Representation:

The connection to Hankel functions is given by:

$$K_0(z) = \frac{i\pi}{2} H_0^{(1)}(iz), \qquad H_0^{(1)}(iz) = -\frac{2i}{\pi} K_0(z)$$

$$\int_{-\infty}^{\infty} \frac{e^{-\sqrt{k_x^2+\alpha^2}|z|}}{\sqrt{k_x^2+\alpha^2}} e^{ik_x x} dk_x = i\pi H_0^{(1)}(i\kappa_s\sqrt{x^2+z^2})$$

so the same integral may be written $I = i\pi H_0^{(1)}(i\kappa_s\sqrt{x^2+z^2})$. where $H_0^{(1)}$ is the Hankel function of the first kind. However, for decaying (non-oscillatory) solutions, the correct representation is the following:

$$I = 2K_0\left(\kappa_s\sqrt{x^2+z^2}\right)$$

Thus, the inverse Fourier transform yields:

$$\tilde{u}_y(x,z,s) = \frac{A}{4\pi\rho(\beta^2+\gamma s)} \cdot 2K_0\left(\kappa_s\sqrt{x^2+z^2}\right)$$

Simplify, writing $R = \sqrt{x^2+z^2}$:

$$\tilde{u}_y(x,z,s) = \frac{A}{2\pi\rho(\beta^2+\gamma s)} K_0\left(\kappa_s R\right)$$

where:

$$\kappa_s = \sqrt{\frac{s^2}{\beta^2+\gamma s}} = \frac{s}{\sqrt{\beta^2+\gamma s}}$$

Therefore the *exact closed-form Laplace-domain Green's function* is

$$\boxed{\tilde{u}_y(x,z,s) = \frac{A}{2\pi\rho(\beta^2+\gamma s)}\, K_0\left(\frac{s\,R}{\sqrt{\beta^2+\gamma s}}\right), \qquad R = \sqrt{x^2+z^2}} \tag{12}$$

This is a complete, correct solution in the Laplace domain. It remains only to invert it to time.

3.4.1 Integral Manipulation (CDH Substitution)

To prepare for the Cagniard–De Hoop inversion, substitute $k_x = isp$ into the inverse FT integral. This maps the real k_x-axis onto the imaginary p-axis and parametrises the solution in the slowness variable p (which will later be deformed onto the Cagniard path).

$$\tilde{u}_y(x,z,s) = \mathcal{F}^{-1}\{\hat{\tilde{u}}_y(k_x,z,s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\tilde{u}}_y(k_x,z,s)e^{ik_x x}\,dk_x$$

Under $k_x = isp$ the mapping is $p = -ik_x/s$, so when k_x runs from $-\infty$ to $+\infty$, p runs from $+i\infty$ to $-i\infty$, and $dk_x = is\,dp$. Thus

$$\tilde{u}_y(x,z,s) = \frac{1}{2\pi}\int_{+i\infty}^{-i\infty} \hat{\tilde{u}}_y(isp,z,s)\,e^{i(isp)x}\,(is\,dp).$$

The Fourier-Laplace solution, equation (11), before the inverse Fourier transform was:

$$\hat{\tilde{u}}_y(k_x,z,s) = \frac{A}{2\rho(\beta^2+\gamma s)}\cdot\frac{e^{-\kappa|z|}}{\kappa}, \quad \kappa = \sqrt{k_x^2 + \frac{s^2}{\beta^2+\gamma s}}$$

Substitute $k_x = isp$:

$$\kappa = \sqrt{(isp)^2 + \frac{s^2}{\beta^2+\gamma s}} = \sqrt{-s^2p^2 + \frac{s^2}{\beta^2+\gamma s}} = s\sqrt{\frac{1}{\beta^2+\gamma s} - p^2}$$

Notation: define the *vertical slowness*

$$\nu(p,s) \equiv \sqrt{\frac{1}{\beta^2+\gamma s} - p^2}$$

so that $\kappa = s\nu$. (Note: ν depends on the Laplace variable s through the effective wave-speed $c_s = \sqrt{\beta^2+\gamma s}$)

$$\hat{\tilde{u}}_y(isp,z,s) = \frac{A}{2\rho(\beta^2+\gamma s)}\cdot\frac{e^{-s\nu|z|}}{s\nu}$$

The inverse Fourier transform becomes:

$$\tilde{u}_y(x,z,s) = \frac{1}{2\pi}\int_{+i\infty}^{-i\infty} \frac{A}{2\rho(\beta^2+\gamma s)}\cdot\frac{e^{-s\nu|z|}}{s\nu}\,e^{-spx}\,(is\,dp)$$

Simplify ($is/s = i$, cancel s):

$$\tilde{u}_y(x,z,s) = \frac{iA}{4\pi\rho(\beta^2+\gamma s)}\int_{+i\infty}^{-i\infty} \frac{e^{-s(px+\nu|z|)}}{\nu}\,dp$$

Flipping the limits introduces a minus sign:

$$\boxed{\tilde{u}_y(x,z,s) = \frac{A}{4\pi\rho(\beta^2+\gamma s)}\int_{+i\infty}^{-i\infty}\frac{ie^{-s(px+\nu|z|)}}{\nu}\,dp = \frac{-A}{4\pi\rho(\beta^2+\gamma s)}\int_{-i\infty}^{+i\infty}\frac{ie^{-s(px+\nu|z|)}}{\nu}\,dp}$$

$$\boxed{\nu = \sqrt{\frac{1}{\beta^2+\gamma s}-p^2}, \quad \beta^2 = \frac{\mu}{\rho}, \quad \gamma = \frac{\zeta}{\rho}}$$

3.5 Step 5: Cagniard Path

The next step is to deform the integration contour in the complex p-plane such that the exponent $-s(px+\nu|z|)$ becomes real and positive, simplifying the inversion.

The integral over p running from $-i\infty$ to $+i\infty$ can first be converted using the symmetry of the integrand on the imaginary p-axis. Setting $p = iq$ (real q), one verifies that $\nu(iq) = \sqrt{c_s^{-2}+q^2}$ is real and that the integrand is even in q in its real part and odd in its imaginary part, giving:

$$\int_{-i\infty}^{+i\infty}\frac{-ie^{-s(px+\nu|z|)}}{\nu}\,dp = 2\mathrm{Im}\left\{\int_0^{+i\infty}\frac{e^{-s(px+\nu|z|)}}{\nu}\,dp\right\}$$

Therefore:

$$\tilde{u}_y(x,z,s) = \frac{A}{2\pi\rho(\beta^2+\gamma s)}\mathrm{Im}\left\{\int_0^{+i\infty}\frac{e^{-s(px+\nu|z|)}}{\nu}\,dp\right\} \tag{13}$$

Important Explanation: In the lossless case ($\gamma = 0$) one has $\nu = \sqrt{\beta^{-2}-p^2}$, which is *independent of* s, so the exponent is exactly $-s\,w(p)$ with $w(p) = px+\nu|z|$, and the Cagniard map $w(p) = \tau$ yields the time-domain answer by inspection.
For Kelvin–Voigt, $\nu = \nu(p,s)$ *depends on* s. Consequently $px+\nu|z|$ is *not* of the form $w(p)$ alone, the Cagniard contour itself becomes s-dependent, and the slowness integral is *not* a forward Laplace transform. The inversion therefore **cannot be completed in elementary closed form by the standard Cagniard recipe**.
This is a structural feature of dissipative media: the classical Cagniard–De Hoop technique applies to piecewise-homogeneous, loss-free media; with dissipation the inverse transform is generally left as a (single) integral evaluated analytically only in special limits or numerically otherwise.

3.5.1 The Cagniard Path Equation

Define the Cagniard travel-time variable:

$$\tau = px + \nu|z|$$

Isolate the term containing ν and square both sides:

$$\begin{aligned} \tau - px &= \nu|z| \\ (\tau - px)^2 &= \nu^2 z^2 \\ \tau^2 - 2\tau px + p^2x^2 &= \nu^2 z^2 \end{aligned}$$

Substitute the definition $\nu^2 = \frac{1}{\beta^2+\gamma s} - p^2$:

$$\tau^2 - 2\tau px + p^2x^2 = \left(\frac{1}{\beta^2 + \gamma s} - p^2\right) z^2$$

Expand the right side:

$$\tau^2 - 2\tau px + p^2x^2 = \frac{z^2}{\beta^2 + \gamma s} - p^2z^2$$

Group all terms on the left side to form a quadratic equation in terms of p:

$$p^2(x^2 + z^2) - 2\tau xp + \tau^2 - \frac{z^2}{\beta^2 + \gamma s} = 0$$

Apply the substitution $R^2 = x^2 + z^2$:

$$R^2p^2 - (2\tau x)p + \left(\tau^2 - \frac{z^2}{\beta^2 + \gamma s}\right) = 0$$

Apply the quadratic formula $p = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$:

$$p = \frac{2\tau x \pm \sqrt{(-2\tau x)^2 - 4(R^2)\left(\tau^2 - \frac{z^2}{\beta^2+\gamma s}\right)}}{2R^2}$$

Let's simplify the discriminant (Δ) under the square root:

$$\Delta = 4\tau^2x^2 - 4R^2\tau^2 + \frac{4R^2z^2}{\beta^2 + \gamma s}$$

Substitute $R^2 = x^2 + z^2$ exclusively into the middle term to cancel the x^2 components:

$$\Delta = 4\left[\tau^2 x^2 - (x^2 + z^2)\tau^2 + \frac{R^2 z^2}{\beta^2 + \gamma s}\right]$$
$$\Delta = 4\left[\tau^2 x^2 - \tau^2 x^2 - \tau^2 z^2 + \frac{R^2 z^2}{\beta^2 + \gamma s}\right]$$
$$\Delta = 4\left[-\tau^2 z^2 + \frac{R^2 z^2}{\beta^2 + \gamma s}\right]$$

Factor out z^2:

$$\Delta = 4z^2\left(\frac{R^2}{\beta^2 + \gamma s} - \tau^2\right)$$

Substitute the simplified discriminant back into the quadratic formula for p:

$$p = \frac{2\tau x \pm \sqrt{4z^2\left(\frac{R^2}{\beta^2+\gamma s} - \tau^2\right)}}{2R^2}$$

Take the square root of $4z^2$ (which yields $2z$) and divide the numerator and denominator by 2:

$$p(\tau) = \frac{\tau x \pm z\sqrt{\frac{R^2}{\beta^2+\gamma s} - \tau^2}}{R^2}$$

Applying the quadratic formula yields the Cagniard path:

$$\boxed{p(\tau) = \frac{x\tau \pm |z|\sqrt{\dfrac{R^2}{\beta^2 + \gamma s} - \tau^2}}{R^2}}$$

Let $T_s = R/\sqrt{\beta^2 + \gamma s}$ denote the *Laplace-domain arrival time*. Then:

- For $\tau < T_s$: $p(\tau)$ is real (the path lies on the real p-axis).
- For $\tau \geq T_s$: $p(\tau)$ becomes complex, $p(\tau) = \frac{x\tau}{R^2} + i\frac{|z|}{R^2}\sqrt{\tau^2 - T_s^2}$, and this is the *Cagniard path.*

3.5.2 Derivative of p Along the Cagniard Path

Differentiate $\tau = px + \nu|z|$ implicitly with respect to τ:

$$1 = x\frac{dp}{d\tau} + |z|\frac{d\nu}{d\tau}, \qquad \text{where} \quad \frac{d\nu}{d\tau} = \frac{-p}{\nu}\frac{dp}{d\tau}$$

Solving:

$$1 = \frac{dp}{d\tau}\left(x - \frac{p|z|}{\nu}\right)$$

$$\boxed{\frac{dp}{d\tau} = \frac{\nu}{x\nu - p|z|}} \tag{14}$$

Key Algebraic Identity

Taking the square of the denominator in (14):

$$\left(x\nu - p|z|\right)^2 = x^2\nu^2 - 2xp\nu|z| + p^2z^2$$

$$= x^2\left(\frac{1}{\beta^2+\gamma s} - p^2\right) - 2xp\nu|z| + p^2z^2$$

$$= \frac{x^2}{\beta^2+\gamma s} - x^2p^2 - 2xp\nu|z| + p^2z^2$$

$$= -(x^2p^2 - p^2z^2) - 2xp\nu|z| + \frac{x^2}{\beta^2+\gamma s}$$

$$= -\left(\tau^2 - \frac{z^2}{\beta^2+\gamma s} - 2px\nu|z|\right) - 2xp\nu|z| + \frac{x^2}{\beta^2+\gamma s}$$

$$= -\tau^2 + \frac{z^2}{\beta^2+\gamma s} + \frac{x^2}{\beta^2+\gamma s}$$

$$= \frac{x^2+z^2}{\beta^2+\gamma s} - \tau^2 = \frac{R^2}{\beta^2+\gamma s} - \tau^2$$

$$\boxed{(x\nu - p|z|)^2 = \frac{R^2}{\beta^2+\gamma s} - \tau^2}$$

Therefore:

$$x\nu - p|z| = \begin{cases} +\sqrt{\dfrac{R^2}{\beta^2+\gamma s} - \tau^2}, & \tau < T_s \,=\, \tau < \dfrac{R}{\sqrt{\beta^2+\gamma s}} \\ -i\sqrt{\tau^2 - \dfrac{R^2}{\beta^2+\gamma s}}, & \tau \geq T_s \,=\, \tau \geq \dfrac{R}{\sqrt{\beta^2+\gamma s}} \end{cases} \tag{15}$$

The sign in the second case is determined by the Cagniard branch chosen. Remember that the integral to be deformed runs from $p = 0$ to $p = +i\infty$ (positive imaginary axis, see equation (13)), so we must use the *upper* Cagniard branch, for which $\mathrm{Im}(p) > 0$:

$$p_+(\tau) = \frac{x\tau}{R^2} + i\frac{|z|\sqrt{\tau^2 - T_s^2}}{R^2}$$

Direct calculation confirms that for p_+ the denominator is *purely imaginary with a negative imaginary part*:

$$\boxed{x\nu - p|z| = -i\sqrt{\tau^2 - T_s^2}, \qquad \tau \geq T_s \quad \text{(upper branch } p_+\text{).}}$$

3.6 Step 6: CDH Contour Deformation and Identification as Laplace Transform

Recall from equation (13) Step 5:

$$\tilde{u}_y(x, z, s) = \frac{A}{2\pi\rho(\beta^2 + \gamma s)} \,\mathrm{Im}\left\{\int_0^{+i\infty} \frac{e^{-s(px+\nu|z|)}}{\nu}\, dp\right\}$$

The Cagniard path $\mathcal{C}$ is parametrised by real $\tau \geq T_s$:

$$p(\tau) = \frac{x\tau}{R^2} + i\frac{|z|\sqrt{\tau^2 - T_s^2}}{R^2}, \qquad T_s = \frac{R}{\sqrt{\beta^2 + \gamma s}}$$

3.6.1 Contour Deformation

The original contour runs along the imaginary p-axis from 0 to $+i\infty$. By Cauchy's theorem (no singularities are enclosed by the deformation), this may be replaced by:

1. A segment along the *real* p-axis from $p = 0$ to $p = p(T_s) = xT_s/R^2$. On this real segment ν is real and positive, so the integrand is *real*; its imaginary part is zero.
2. The Cagniard path $\mathcal{C}$ from $\tau = T_s$ to $\tau = +\infty$.

Hence only the Cagniard path contributes to the imaginary part:

$$\mathrm{Im}\left\{\int_0^{+i\infty} \frac{e^{-s(px+\nu|z|)}}{\nu}\, dp\right\} = \mathrm{Im}\left\{\int_{T_s}^{\infty} \frac{e^{-s\tau}}{\nu}\frac{dp}{d\tau}\, d\tau\right\}$$

where we have used $px + \nu|z| = \tau$ on $\mathcal{C}$ so $e^{-s(px+\nu|z|)} = e^{-s\tau}$ is real.

3.6.2 Evaluating $\mathrm{Im}\{(1/\nu)\, dp/d\tau\}$ on the Cagniard Path

From (14):

$$\frac{1}{\nu}\frac{dp}{d\tau} = \frac{1}{x\nu - p|z|}$$

On the upper Cagniard path ($\tau \geq T_s$) we showed $x\nu - p|z| = -i\sqrt{\tau^2 - T_s^2}$, so:

$$\left.\frac{1}{\nu}\frac{dp}{d\tau}\right|_{\mathcal{C}} = \frac{1}{-i\sqrt{\tau^2 - T_s^2}} = \frac{i}{\sqrt{\tau^2 - T_s^2}}$$

which is *purely imaginary with positive imaginary part.* Therefore:

$$\mathrm{Im}\left\{\frac{1}{\nu}\frac{dp}{d\tau}\right\}_{\mathcal{C}} = \mathrm{Im}\left\{\frac{i}{\sqrt{\tau^2 - T_s^2}}\right\} = \frac{1}{\sqrt{\tau^2 - T_s^2}}$$

3.6.3 The CDH Integral Representation

Combining all the above:

$$\tilde{u}_y(x, z, s) = \frac{A}{2\pi\rho(\beta^2 + \gamma s)} \int_{T_s}^{\infty} \frac{e^{-s\tau}}{\sqrt{\tau^2 - T_s^2}}\, d\tau$$

Consistency check. The standard integral representation of the modified Bessel function is:

$$K_0(as) = \int_a^{\infty} \frac{e^{-s\tau}}{\sqrt{\tau^2 - a^2}}\, d\tau, \qquad \mathrm{Re}(s) > 0,\ a > 0$$

See 3.961.2 of [1]

Setting, $a = T_s = R/\sqrt{\beta^2 + \gamma s}$ we immediately recover the result from Step 4 (equation (12)):

$$\boxed{\tilde{u}_y(x, z, s) = \frac{A}{2\pi\rho(\beta^2 + \gamma s)}\, K_0\!\left(\frac{sR}{\sqrt{\beta^2 + \gamma s}}\right)}$$

confirming the two independent derivations (direct K_0 evaluation and Cagniard–De Hoop contour deformation) are consistent.

3.7 Step 7: Inverse Laplace Transform and Time-Domain Solution

The exact causal full time-domain Green's function solution is recovered by the Bromwich inversion integral:

$$u_y(x,z,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{A}{2\pi\rho(\beta^2+\gamma s)}\, K_0\!\left(\frac{sR}{\sqrt{\beta^2+\gamma s}}\right) e^{st}\, ds, \qquad c>0$$

3.7.1 (a) Elastic Limit ($\gamma \to 0$): The Exact Closed-Form Time-Domain Green's Function Solution

When $\gamma \to 0$ the Kelvin–Voigt model reduces to the purely elastic medium with shear-wave speed β. Setting $\gamma = 0$:

- The effective wave speed $\sqrt{\beta^2+\gamma s} \to \beta$ (constant, s-independent).
- $T_s \to T_0 = R/\beta$ (constant arrival time).
- The CDH integral becomes:

$$\tilde{u}_y^{\text{elastic}}(x,z,s) = \frac{A}{2\pi\rho\beta^2}\int_{R/\beta}^{\infty} \frac{e^{-s\tau}}{\sqrt{\tau^2 - R^2/\beta^2}}\, d\tau = \frac{A}{2\pi\rho\beta^2}\, K_0\!\left(\frac{sR}{\beta}\right)$$

The lower limit $T_0 = R/\beta$ is now *independent of* s, so the integral is a genuine one-sided Laplace transform. Using the standard pair

$$\mathcal{L}\left\{\frac{H(t-\tau_0)}{\sqrt{t^2-\tau_0^2}}\right\}(s) = K_0(s\,\tau_0), \qquad \tau_0 = \frac{R}{\beta}; \qquad \tau_0 > 0$$

we recover the classical 2D SH Green's function exactly:

$$\boxed{u_y^{\text{elastic}}(x,z,t) = \frac{A}{2\pi\rho\beta^2}\,\frac{H\!\left(t-\dfrac{R}{\beta}\right)}{\sqrt{t^2-\dfrac{R^2}{\beta^2}}}, \qquad R=\sqrt{x^2+z^2}} \tag{16}$$

where $R = \sqrt{x^2+z^2}$ and $H(\cdot)$ is the Heaviside step function. This is the well-known 2D SH-wave Green's function for a homogeneous elastic medium: the signal arrives sharply at $t = R/\beta$ and decays as t^{-1} for large t.

3.7.2 (b) Viscoelastic Case ($\gamma > 0$): exact single-integral (Bromwich) representation

For $\gamma > 0$ the effective arrival time $T_s = R/\sqrt{\beta^2+\gamma s}$ depends on s, so from Cagniard–de Hoop (CDH) method, the Laplace-transformed displacement is:

$$\tilde{u}_y(x,z,s) = \frac{A}{2\pi\rho(\beta^2+\gamma s)}\int_{T_s}^{\infty} \frac{e^{-s\tau}}{\sqrt{\tau^2-T_s^2}}\, d\tau, \qquad T_s = \frac{R}{\sqrt{\beta^2+\gamma s}}$$

valid for $\mathrm{Re}(s) > 0$, $\gamma > 0$, with $R = \sqrt{x^2 + z^2}$.

is *not* a Laplace transform of a fixed function of τ, because both the prefactor and the integration limits depend on s. Using the standard integral representation described above, we have:

$$\tilde{u}_y(x, z, s) = \frac{A}{2\pi\rho(\beta^2 + \gamma s)} K_0\left(\frac{sR}{\sqrt{\beta^2 + \gamma s}}\right) \tag{17}$$

The exact causal time-domain solution is the inverse Laplace transform of (17) obtained from the Bromwich integral:

$$u_y(x, z, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{u}_y(x, z, s)\, e^{st}\, ds, \qquad c > 0$$

Substituting (17):

$$\boxed{u_y(x, z, t) = \frac{A}{2\pi\rho} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\beta^2 + \gamma s} K_0\left(\frac{sR}{\sqrt{\beta^2 + \gamma s}}\right) e^{st}\, ds} \tag{18}$$

This is the **exact Bromwich representation** – the fundamental starting point for numerical or analytical evaluation. It is exact, causal, and valid for $\gamma \geq 0$ (including elastic when $\gamma = 0$, though then there's a branch point at $s = 0$). For $\gamma > 0$, the integrand is analytic except for branch cuts and possibly essential singularities (as noted at $s = -\beta^2/\gamma$). It may be reduced to a *single real integral* suitable for computation. Using the Cagniard-type branch relation (15) in (13), the time-domain field is

$$\boxed{u_y(x, z, t) = \frac{A}{2\pi\rho} \mathcal{L}^{-1}\left\{ \frac{1}{\beta^2 + \gamma s} \int_{\tau_s}^{\infty} \frac{e^{-s\,t'}}{\sqrt{t'^2 - \dfrac{R^2}{\beta^2 + \gamma s}}} H(t' - \tau_s)\, dt' \right\}, \quad \tau_s = \frac{R}{\sqrt{\beta^2 + \gamma s}}} \tag{19}$$

Equivalently, the dissipative solution is the elastic kernel (16) convolved with the inverse transform of the s-dependent dispersion factor; in practice (18) is evaluated by a numerical Laplace inversion (e.g. Talbot's method or fixed-Bromwich quadrature), which is the accepted state of the art for Kelvin–Voigt media.

3.7.3 Frequency-Domain (Inverse Fourier Transform) Representation

The Laplace transform $\tilde{u}_y(s)$ possesses an *essential singularity* at $s = -\beta^2/\gamma$: as $s \to -\beta^2/\gamma$ from the right the argument of K_0 diverges along the negative real axis and K_0

grows exponentially, so the Bromwich contour cannot be closed to the left past that point.

The time-domain result is obtained by moving the Bromwich contour to the imaginary axis ($c \to 0^+$), giving the equivalent *inverse Fourier transform*:

Setting $s = i\omega + \varepsilon$ with $\varepsilon \to 0^+$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega t}d\omega$$

when $F(s)$ is analytic on and to the right of the imaginary axis and decays sufficiently.

Here:

$$F(s) = \frac{A}{2\pi\rho}\cdot\frac{1}{\beta^2+\gamma s}\,K_0\left(\frac{sR}{\sqrt{\beta^2+\gamma s}}\right)$$

Thus:

$$F(i\omega) = \frac{A}{2\pi\rho}\cdot\frac{1}{\beta^2+i\gamma\omega}\,K_0\left(\frac{i\omega R}{\sqrt{\beta^2+i\gamma\omega}}\right)$$

Multiplying by $1/(2\pi)$ from the Fourier inversion formula gives:

$$\boxed{u_y(x,z,t) = \frac{A}{4\pi^2\rho}\int_{-\infty}^{\infty}\frac{K_0\left(\dfrac{i\omega R}{\sqrt{\beta^2+i\gamma\omega}}\right)}{\beta^2+i\gamma\omega}\,e^{i\omega t}\,d\omega}$$

For $\omega > 0$, the identity $K_0(iz) = \frac{i\pi}{2}H_0^{(1)}(z)$ (with $\mathrm{Im}(z) > 0$) and the complex wavenumber $k(\omega) = \omega/c(\omega)$, $c(\omega) = \sqrt{(\mu + i\omega\eta)/\rho}$, give:

$$K_0\left(\frac{i\omega R}{\sqrt{\beta^2+i\gamma\omega}}\right) = \frac{i\pi}{2}\,H_0^{(1)}\big(k(\omega)\,R\big)$$

confirming the standard outgoing cylindrical-wave form of the 2D Green's function for an attenuating-dispersive medium. The integral is evaluated numerically.

General Notes:

- Because $|c(\omega)| \to \infty$ as $\omega \to \infty$, the Kelvin–Voigt model is *not strictly causal* at infinite frequency; consequently $u_y > 0$ for all $t > 0$ (no sharp wave front).
- The imaginary part of $k(\omega)$ provides spatial attenuation; the real part of $c(\omega)$ varies with frequency (dispersion), smearing the waveform relative to the elastic case.
- As $\gamma \to 0$: $c(\omega) \to \beta$, $k(\omega) \to \omega/\beta$, and the integral reduces to the elastic closed form u_y^{elastic}.

Why the CDH form cannot be directly inverted

In the elastic case ($\gamma = 0$), $T_s = R/\beta$ is constant, and (18) reduces to a known inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{1}{\beta^2}K_0\left(\frac{sR}{\beta}\right)\right\} = \frac{1}{\beta^2}\frac{H(t-R/\beta)}{\sqrt{t^2-(R/\beta)^2}}$$

For $\gamma > 0$ no elementary closed form exists. For $\gamma > 0$, T_s depends on s, so the representation

$$\tilde{u}_y = \frac{A}{2\pi\rho(\beta^2+\gamma s)}\int_{T_s}^{\infty}\frac{e^{-s\tau}}{\sqrt{\tau^2-T_s^2}}d\tau$$

is **not** a simple Laplace transform of a fixed τ-function – the s-dependence inside the integral and the limit T_s prevent direct term-by-term inversion. Hence, one must invert the compact K_0 form (18) directly.

3.8 Summary of Analytical Results

Domain	**Elastic** ($\gamma = 0$)	**Kelvin–Voigt** ($\gamma > 0$)
Laplace	$\frac{A}{2\pi\rho\beta^2}K_0\left(\frac{sR}{\beta}\right)$	$\frac{A}{2\pi\rho(\beta^2+\gamma s)}K_0\left(\frac{sR}{\sqrt{\beta^2+\gamma s}}\right)$
CDH integral	$\frac{A}{2\pi\rho\beta^2}\int_{R/\beta}^{\infty}\frac{e^{-st}}{\sqrt{t^2-R^2/\beta^2}}dt$	$\frac{A}{2\pi\rho(\beta^2+\gamma s)}\int_{T_s}^{\infty}\frac{e^{-st}}{\sqrt{t^2-T_s^2}}dt$
Time domain (Brownwich)	$\frac{A}{2\pi\rho\beta^2}\frac{H(t-R/\beta)}{\sqrt{t^2-R^2/\beta^2}}$	$u_y(x,z,t) = \frac{A}{2\pi\rho}\cdot\frac{1}{2\pi i}\times \int_{c-i\infty}^{c+i\infty}\frac{1}{\beta^2+\gamma s}K_0\left(\frac{sR}{\sqrt{\beta^2+\gamma s}}\right)e^{st}\,ds$
Frequency representation	-	$\frac{A}{4\pi^2\rho}\int_{-\infty}^{\infty}\frac{K_0\left(\frac{i\omega R}{\sqrt{\beta^2+i\gamma\omega}}\right)}{\beta^2+i\gamma\omega}e^{i\omega t}d\omega$

where $R = \sqrt{x^2+z^2}$, $\beta^2 = \mu/\rho$, $\gamma = \zeta/\rho$, and $T_s = R/\sqrt{\beta^2+\gamma s}$.

References

[1] Izrail Solomonovich Gradshteyn and Iosif Moiseevich Ryzhik. *Table of integrals, series, and products*. Academic press, 2014.